Equation of equilibrium (4.51) reduces to

$$\sigma \frac{d(hr)}{dr} - \sigma h + \rho\omega^2 hr^2 = 0$$

where σ is the constant value of the stress. Carrying out the differentiation and combining, we find

$$\frac{1}{h}\frac{dh}{dr} = -\frac{\rho\omega^2}{\sigma} r$$

which, upon integration, gives

$$\log h = -\frac{\rho\omega^2}{2\sigma} r^2 + c_3$$

or

$$h = e^{-(\rho\omega^2 r^2/2\sigma)+c_3} = e^{c_3}e^{-(\rho\omega^2 r^2/2\sigma)} = c_4 e^{-(\rho\omega^2 r^2/2\sigma)} \tag{4.56}$$

where c_3 and c_4 are constants.

4.8. Thermal Stresses in Thin Disks and Long Cylinders. In previous discussions we have assumed that the state of strain is due solely to the applied forces. There are other causes because of which stresses may be set up in an elastic body. One of them is the unequal heating of different parts of the body. With a few exceptions, the elements of a body expand as the temperature is increased. If the element is allowed to expand freely, the body will be strained but there will not be any stress due to such an expansion. However, if the temperature rise in the body is not uniform and the body is continuous, the expansion of the elements cannot proceed freely and *thermal stresses* are produced. The problem of determining the thermal stresses in an elastic body due to a given temperature distribution finds many practical applications in machine design, such as in the design of steam and gas turbines and internal-combustion engines.

Let us consider first an unstrained elastic body with a uniform temperature T_0. Now imagine that the body is heated to some temperature T above T_0. The body will be stressed if T varies from point to point in the body. The strain of an element may be considered as consisting of two parts. One part is due to the expansion of the element because of the change of its temperature. If α is the *coefficient of linear expansion* of the material, which is defined as the change in length per unit length per degree rise in temperature, this part of longitudinal strain will be αT. There will be no shearing strains produced, because the expansion of a small element, due to change of temperature, will not produce angular distortion in an isotropic material. If the element is allowed to expand freely, this is the only component of strain and the element will not be stressed. Now, if the element is not allowed to expand freely, stresses will be produced and the total strain of the element must be the sum of that part due to the stresses and that due to the change of the temperature.

Referring to cartesian coordinates, if the stress components at the point are $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$, the strain components are therefore

$$\begin{aligned} \epsilon_x &= \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] + \alpha T & \gamma_{xy} &= \frac{1}{G}\tau_{xy} \\ \epsilon_y &= \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] + \alpha T & \gamma_{yz} &= \frac{1}{G}\tau_{yz} \\ \epsilon_z &= \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] + \alpha T & \gamma_{zx} &= \frac{1}{G}\tau_{zx} \end{aligned} \tag{4.57}$$

From the above formulas, we obtain the relations between stress and strain as follows,

$$\begin{aligned} \sigma_x &= \lambda e + \frac{E}{1+\nu}\epsilon_x - \frac{\alpha E T}{1-2\nu} \\ \sigma_y &= \lambda e + \frac{E}{1+\nu}\epsilon_y - \frac{\alpha E T}{1-2\nu} \\ \sigma_z &= \lambda e + \frac{E}{1+\nu}\epsilon_z - \frac{\alpha E T}{1-2\nu} \end{aligned} \tag{4.58}$$

where λ is equal to $\nu E/(1+\nu)(1-2\nu)$ and $e = \epsilon_x + \epsilon_y + \epsilon_z$. The relations between τ and γ are the same as in the case when there are no thermal strains.

Now let us consider a thin circular disk with uneven temperature distribution. Assume the temperature T is a function of the radial distance r only. We have a case of plane stress with rotational symmetry. In terms of cylindrical coordinates, we find, from (4.57),

$$\epsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta) + \alpha T \qquad \epsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) + \alpha T \tag{4.59}$$

The equilibrium equation

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

is identically satisfied if we introduce the stress function ψ such that

$$\sigma_r = \frac{\psi}{r} \qquad \sigma_\theta = \frac{d\psi}{dr} \tag{4.60}$$

Substituting (4.59) and (4.60) into the compatibility equation (4.39)

$$r\frac{d\epsilon_\theta}{dr} + \epsilon_\theta - \epsilon_r = 0$$

and simplifying, we find

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} - \frac{\psi}{r^2} = -\alpha E\frac{dT}{dr}$$

or

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r\psi)\right] = -\alpha E\frac{dT}{dr} \tag{4.61}$$

This equation can be easily integrated, and the solution is

$$\psi = -\frac{\alpha E}{r}\int_a^r Tr\,dr + \frac{c_1 r}{2} + \frac{c_2}{r} \tag{4.62}$$

where the lower limit a in the integral can be chosen arbitrarily. For a disk with a hole, it may be the inner radius. For a solid disk we may take it as zero.

The stress components can now be found by substituting (4.62) into formulas (4.60). Hence

$$\sigma_r = -\frac{\alpha E}{r^2}\int_a^r Tr\,dr + \frac{c_1}{2} + \frac{c_2}{r^2}$$

$$\sigma_\theta = \alpha E\left(-T + \frac{1}{r^2}\int_a^r Tr\,dr\right) + \frac{c_1}{2} - \frac{c_2}{r^2}$$

For a solid disk, we must have finite stresses at the center, and therefore c_2 must be zero. If there is no external force applied at the boundary, $\sigma_r = 0$ at $r = b$. It follows then that

$$c_1 = \frac{2\alpha E}{b^2}\int_0^b Tr\,dr$$

The stress components are

$$\sigma_r = \alpha E\left(\frac{1}{b^2}\int_0^b Tr\,dr - \frac{1}{r^2}\int_0^r Tr\,dr\right)$$
$$\sigma_\theta = \alpha E\left(-T + \frac{1}{b^2}\int_0^b Tr\,dr + \frac{1}{r^2}\int_0^r Tr\,dr\right) \tag{4.63}$$

Consider, as an example, a thin disk which receives heat over its faces and rejects it at its circumference in such a way that the temperature at any point in the disk is essentially uniform through the thickness. If T_0 is the temperature at the edge of the disk and T_1 is the temperature at the center, the temperature rise at a radius r is given by

$$T = (T_1 - T_0) - (T_1 - T_0)\frac{r^2}{b^2}$$

Substituting the expression of T given by the above formula into Eqs. (4.63) and integrating, we obtain

$$\sigma_r = -\frac{1}{4}\alpha E(T_1 - T_0)\left(1 - \frac{r^2}{b^2}\right) \qquad \sigma_\theta = -\frac{1}{4}\alpha E(T_1 - T_0)\left(1 - \frac{3r^2}{b^2}\right)$$

If there is a circular hole of radius a at the center of the disk and the edges are free of external forces, we have

$$\sigma_r = 0 \qquad \text{at } r = b \text{ and } r = a$$

Then

$$\frac{c_1}{2} + \frac{c_2}{b^2} = \frac{\alpha E}{b^2}\int_a^b Tr\,dr \qquad \frac{c_1}{2} + \frac{c_2}{a^2} = 0$$

from which it follows that

$$\frac{c_1}{2} = \frac{\alpha E}{b^2 - a^2}\int_a^b Tr\,dr \qquad c_2 = -\frac{a^2\alpha E}{b^2 - a^2}\int_a^b Tr\,dr$$

and
$$\sigma_r = \alpha E\left[-\frac{1}{r^2}\int_a^r Tr\,dr + \frac{1}{b^2 - a^2}\int_a^b Tr\,dr - \frac{a^2}{r^2(b^2 - a^2)}\int_a^b Tr\,dr\right]$$
$$\sigma_\theta = \alpha E\left[-T + \frac{1}{r^2}\int_a^r Tr\,dr + \frac{1}{b^2 - a^2}\int_a^b Tr\,dr - \frac{a^2}{r^2(b^2 - a^2)}\int_a^b Tr\,dr\right] \tag{4.64}$$

Let us now consider the thermal stresses in a long circular cylinder with a temperature distribution symmetrical about its axis. If the ends of the cylinder are restrained in such a way that $\epsilon_z = 0$, we have a plane-strain problem. In terms of cylindrical coordinates, the stress-strain relations are

$$\epsilon_r = \frac{1}{E}[\sigma_r - \nu(\sigma_\theta + \sigma_z)] + \alpha T$$
$$\epsilon_\theta = \frac{1}{E}[\sigma_\theta - \nu(\sigma_r + \sigma_z)] + \alpha T \tag{4.65}$$
$$\epsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_r + \sigma_\theta)] + \alpha T$$

In the case of plain strain, $\epsilon_z = 0$, and the third equation of (4.65) gives

$$\sigma_z = \nu(\sigma_r + \sigma_\theta) - \alpha ET \tag{4.66}$$

Substituting (4.66) into the first two equations of (4.65), we obtain

$$\epsilon_r = \frac{1+\nu}{E}[(1-\nu)\sigma_r - \nu\sigma_\theta + \alpha ET]$$
$$\epsilon_\theta = \frac{1+\nu}{E}[(1-\nu)\sigma_\theta - \nu\sigma_r + \alpha ET] \tag{4.67}$$

Let us now substitute (4.67) and (4.60) into the compatibility equation (4.39); after some simplification we obtain

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} - \frac{\psi}{r^2} = -\frac{\alpha E}{1-\nu}\frac{dT}{dr} \tag{4.68}$$

Comparing with Eq. (4.61), we find that these two equations are the same except the coefficient of dT/dr. The solution is therefore

$$\psi = -\frac{\alpha E}{1-\nu}\frac{1}{r}\int_a^r Tr\,dr + \frac{c_1 r}{2} + \frac{c_2}{r}$$

from which it follows that

$$\sigma_r = -\frac{\alpha E}{1-\nu}\frac{1}{r^2}\int_a^r Tr\,dr + \frac{c_1}{2} + \frac{c_2}{r^2}$$

For a solid cylinder, $c_2 = 0$ so that the stresses in the cylinder will be finite. On the outer surface $r = b$, $\sigma_r = 0$, and we find

$$c_1 = \frac{2\alpha E}{1-\nu}\frac{1}{b^2}\int_0^b Tr\,dr$$

The stress components are therefore

$$\begin{aligned}\sigma_r &= \frac{\alpha E}{1-\nu}\left(\frac{1}{b^2}\int_0^b Tr\,dr - \frac{1}{r^2}\int_0^r Tr\,dr\right)\\ \sigma_\theta &= \frac{\alpha E}{1-\nu}\left(-T + \frac{1}{b^2}\int_0^b Tr\,dr + \frac{1}{r^2}\int_0^r Tr\,dr\right)\end{aligned} \qquad (4.69)$$

and, from Eq. (4.66)

$$\sigma_z = \frac{\alpha E}{1-\nu}\left(\frac{2\nu}{b^2}\int_0^b Tr\,dr - T\right) \qquad (4.70)$$

This is the normal stress distribution which must be applied to keep $\epsilon_z = 0$ throughout. If the cylinder has free ends, we can superpose on it a uniform axial stress $\sigma_z = c_3$ so that the resultant force on the ends is zero. Integrating, we find that the condition

$$\int_0^b \sigma_z 2\pi r\,dr = 0$$

gives

$$c_3 = \frac{2\alpha E}{b^2}\int_0^b Tr\,dr$$

In such a case, we have therefore

$$\sigma_z = \frac{\alpha E}{1-\nu}\left(\frac{2}{b^2}\int_0^b Tr\,dr - T\right) \qquad (4.71)$$

In the case of a circular cylinder with a concentric circular hole, the constants of integration can be determined by the conditions that $\sigma_r = 0$ at $r = b$ and $r = a$. Then,

$$\frac{c_1}{2} + \frac{c_2}{b^2} = \frac{\alpha E}{1-\nu}\frac{1}{b^2}\int_a^b Tr\,dr \qquad \frac{c_1}{2} + \frac{c_2}{a^2} = 0$$

Solving, we have

$$\begin{aligned}\frac{c_1}{2} &= \frac{\alpha E}{1-\nu}\frac{1}{b^2-a^2}\int_a^b Tr\,dr\\ c_2 &= -\frac{\alpha E}{1-\nu}\frac{a^2}{b^2-a^2}\int_a^b Tr\,dr\end{aligned}$$

With these constants, we find

$$\begin{aligned}\sigma_r &= \frac{\alpha E}{1-\nu}\frac{1}{r^2}\left(\frac{r^2-a^2}{b^2-a^2}\int_a^b Tr\,dr - \int_a^r Tr\,dr\right)\\ \sigma_\theta &= \frac{\alpha E}{1-\nu}\frac{1}{r^2}\left(\frac{r^2+a^2}{b^2-a^2}\int_a^b Tr\,dr + \int_a^r Tr\,dr - Tr^2\right)\end{aligned} \qquad (4.72)$$

and if we add to σ_z the uniform stress so that the resultant axial force is zero,

$$\sigma_z = \frac{\alpha E}{1-\nu}\left(\frac{2}{b^2-a^2}\int_a^b Tr\,dr - T\right) \qquad (4.73)$$

If T_1 is the temperature on the inner surface of the cylinder and T_0 is the temperature on the outer surface, in the case of a steady heat flow, the temperature rise T at any distance r from the center is

$$T = \frac{T_1 - T_0}{\log (b/a)}\log\frac{b}{r}$$

Substituting this into Eqs. (4.69) and (4.70), the thermal stresses are

$$\begin{aligned}\sigma_r &= \frac{\alpha E(T_1 - T_0)}{2(1-\nu)\log (b/a)}\left[-\log\frac{b}{r} - \frac{a^2(r^2-b^2)}{r^2(b^2-a^2)}\log\frac{b}{a}\right]\\ \sigma_\theta &= \frac{\alpha E(T_1 - T_0)}{2(1-\nu)\log (b/a)}\left[1 - \log\frac{b}{r} - \frac{a^2(r^2+b^2)}{r^2(b^2-a^2)}\log\frac{b}{a}\right]\\ \sigma_z &= \frac{\alpha E(T_1 - T_0)}{2(1-\nu)\log (b/a)}\left[1 - \log\frac{b}{r} - \frac{2a^2}{b^2-a^2}\log\frac{b}{a}\right]\end{aligned} \qquad (4.74)$$

If T_1 is higher than T_0, the radial stress is compressive at all points and becomes zero at the inner and outer surfaces of the cylinder. The stress components σ_θ and σ_z have their largest numerical values at the inner and outer surfaces of the cylinder.

Problem 1. A thin uniform disk of radius b is enclosed in a heavy ring of the same material into which it just fits when the disk and ring are at a uniform temperature. If the heat is supplied over the faces of the disk and it is rejected at the circumference, the temperature rise at a distance r from the center is given by

$$T = (T_1 - T_0) - (T_1 - T_0)\frac{r^2}{b^2}$$