as indicated in Fig. 3.5b, the solution is exact. Otherwise, the solution is not exact. In any case, recall, however, that Saint-Venant's principle permits us to regard the result as quite accurate for sections away from the ends.

Section 5.4 illustrates the determination of the displacement field after derivation of the curvature–moment relation.

3.6 THERMAL STRESSES

Consider the consequences of increasing or decreasing the *uniform* temperature of an entirely unconstrained elastic body. The resultant expansion or contraction occurs in such a way as to cause a cubic element of the solid to remain cubic, while experiencing changes of length on each of its sides. Normal strains occur in each direction unaccompanied by normal stresses. In addition, there are neither shear strains nor shear stresses. If the body is heated in such a way as to produce a nonuniform temperature field, or if the thermal expansions are prohibited from taking place freely because of restrictions placed on the boundary even if the temperature is uniform, or if the material exhibits anisotropy in a uniform temperature field, thermal stresses will occur. The effects of such stresses can be severe, especially since the most adverse thermal environments are often associated with design requirements involving unusually stringent constraints as to weight and volume. This is especially true in aerospace applications, but is of considerable importance, too, in many everyday machine design applications.

Solution of thermal stress problems requires reformulation of the stress–strain relationships accomplished by superposition of the strain attributable to stress and that due to temperature. For a change in temperature $T(x, y)$, the change of length, δL, of a small linear element of length L in an unconstrained body is $\delta L = \alpha L T$. Here α, a positive number, is termed the coefficient of linear thermal expansion. The thermal strain ε_t associated with the free expansion at a point is then

$$\varepsilon_t = \alpha T \tag{3.22}$$

The total x and y strains, ε_x and ε_y, are obtained by adding to the thermal strains of the type described, the strains due to stress resulting from external forces:

$$\begin{aligned}
\varepsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y) + \alpha T \\
\varepsilon_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x) + \alpha T \qquad \text{(3.23a)} \\
\gamma_{xy} &= \frac{\tau_{xy}}{G}
\end{aligned}$$

In terms of strain components, these expressions become

$$\begin{aligned}
\sigma_x &= \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) - \frac{E\alpha T}{1-\nu} \\
\sigma_y &= \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) - \frac{E\alpha T}{1-\nu} \qquad (3.23b) \\
\tau_{xy} &= G\gamma_{xy}
\end{aligned}$$

Because free thermal expansion results in no angular distortion in an isotropic material, the shearing strain is unaffected, as indicated. Equations (3.23) represent modified strain–stress relations for *plane stress*. Similar expressions may be written for the case of plane strain. The differential equations of equilibrium (3.6) are based on purely mechanical considerations and are unchanged for *thermoelasticity*. The same is true of the strain–displacement relations (2.3) and the compatibility equation (3.8), which are geometrical in character. Thus, for given boundary conditions (expressed either as surface forces or displacements) and temperature distribution, thermoelasticity and ordinary elasticity *differ only* to the extent of the strain–stress relationship.

By substituting the strains given by Eq. (3.23a) into the equation of compatibility (3.8), employing Eq. (3.6) as well, and neglecting body forces, a compatibility equation is derived in terms of stress:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y + \alpha E T) = 0 \qquad (3.24)$$

Introducing Eq. (3.13), we now have

$$\nabla^4\Phi + \alpha E\,\nabla^2 T = 0 \qquad (3.25)$$

This expression is valid for plane strain or plane stress provided that the body forces are negligible.

It has been implicit in treating the matter of thermoelasticity as a superposition problem that the distribution of stress or strain plays a negligible role in influencing the temperature field [Refs. 3.4 and 3.5]. This lack of coupling enables the temperature field to be determined independently of any consideration of stress or strain. If the effect of the temperature distribution on material properties cannot be disregarded, the equations become coupled and analytical solutions are significantly more complex, occupying an area of considerable interest and importance. Numerical solutions can, however, be obtained in a relatively simple manner through the use of finite difference methods.

Example 3.2

A rectangular beam of small thickness t, depth $2h$, and length $2L$ is subjected to an arbitrary variation of temperature throughout its depth, $T = T(y)$. Determine the distribution of stress and strain for the case in which (a) the beam is entirely free of surface forces (Fig. 3.6a), and (b) the beam is held by rigid walls that prevent the x-directed displacement only (Fig. 3.6b).

Solution The beam geometry indicates a problem of plane stress. We begin with the assumptions

$$\sigma_x = \sigma_x(y), \qquad \sigma_y = \tau_{xy} = 0 \tag{a}$$

Direct substitution of Eqs. (a) into Eqs. (3.6) indicates that the equations of equilibrium are satisfied. Equations (a) reduce the compatibility equation (3.24) to the form

$$\frac{d^2}{dy^2}(\sigma_x + \alpha ET) = 0 \tag{b}$$

from which

$$\sigma_x = -\alpha ET + c_1 y + c_2 \tag{c}$$

where c_1 and c_2 are constants of integration. The requirement that faces $y = \pm h$ be free of surface forces is obviously fulfilled by Eq. (b).

(a) The boundary conditions at the end faces are satisfied by determining the constants that assume zero resultant force and moment at $x = \pm L$:

$$\int_{-h}^{h} \sigma_x t\, dy = 0, \qquad \int_{-h}^{h} \sigma_x y t\, dy = 0 \tag{d}$$

Substituting Eq. (c) into Eqs. (d) it is found that $c_1 = (3/2h^3)\int_{-h}^{h} \alpha ETy\, dy$ and $c_2 = (1/2h)\int_{-h}^{h} \alpha ET\, dy$. The normal stress, upon substituting the values of the constants obtained, together with the moment of inertia $I = 2h^3t/3$ and area $A = 2ht$, into Eq. (c) is thus

$$\sigma_x = E\alpha\left[-T + \frac{t}{A}\int_{-h}^{h} T\, dy + \frac{yt}{I}\int_{-h}^{h} Ty\, dy\right] \tag{3.26}$$

The corresponding strains are

$$\varepsilon_x = \frac{\sigma_x}{E} + \alpha T, \qquad \varepsilon_y = -\frac{\nu\sigma_x}{E} + \alpha T, \qquad \gamma_{xy} = 0 \tag{e}$$

The displacements can readily be determined from Eqs. (3.1).

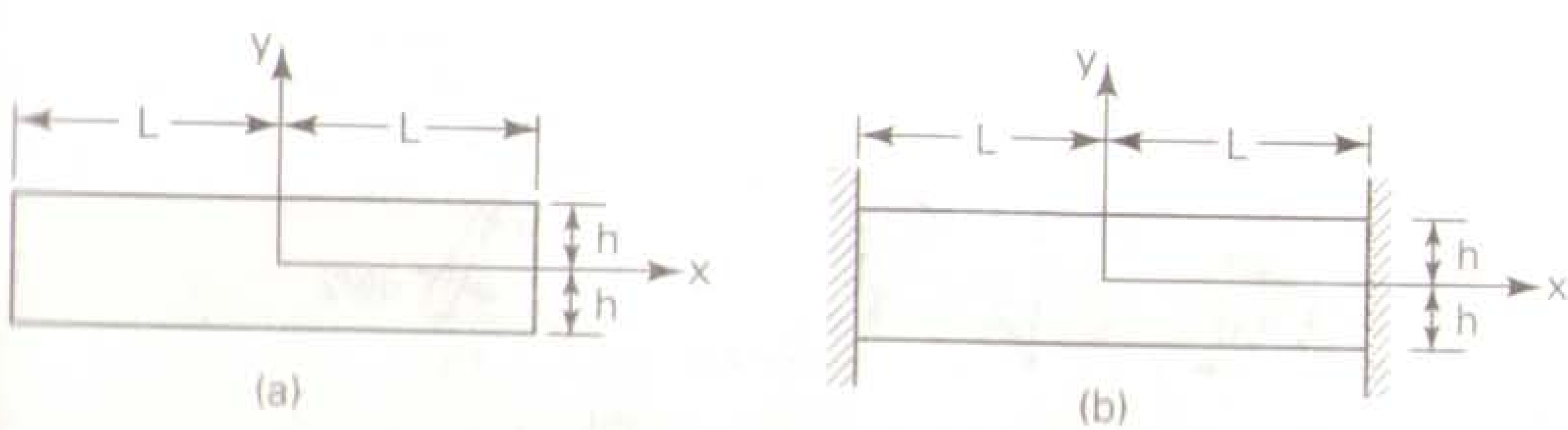

Figure 3.6 Example 3.2. Rectangular beam in plane thermal stress: (a) unsupported; (b) placed between two rigid walls.

From Eq. (3.26), observe that the temperature distribution for T = constant results in zero stress, as expected. Of course, the strains (e) and the displacements will, in this case, not be zero. It is also noted that, when the temperature is symmetrical about the midsurface ($y = 0$), that is, $T(y) = T(-y)$, the final integral in Eq. (3.26) vanishes. For an antisymmetrical temperature distribution about the midsurface, $T(y) = -T(-y)$, and the first integral in Eq. (3.26) is zero.

(b) For the situation described, $\varepsilon_x = 0$ for all y. With $\sigma_y = \tau_{xy} = 0$ and Eq. (c), Eqs. (3.23a) lead to $c_1 = c_2 = 0$, regardless of how T varies with y. Thus,

$$\sigma_x = -E\alpha T \tag{3.27}$$

and

$$\varepsilon_x = \gamma_{xy} = 0, \qquad \varepsilon_y = (1+\nu)\alpha T \tag{f}$$

Note that the axial stress obtained here can be large even for modest temperature changes, as can be verified by substituting properties of a given material.

3.7 BASIC RELATIONS IN POLAR COORDINATES

Geometrical considerations related either to the loading or to the boundary of a loaded system often make it preferable to employ polar coordinates, rather than the Cartesian system used exclusively thus far. In general, polar coordinates are used advantageously where a degree of axial symmetry exists. Examples include a cylinder, a disk, a wedge, a curved beam, and a large thin plate containing a circular hole.

The polar coordinate system (r, θ) and the Cartesian system (x, y) are related by the following expressions (Fig. 3.7a):

$$\begin{aligned} x &= r\cos\theta, \qquad r^2 = x^2 + y^2 \\ y &= r\sin\theta, \qquad \theta = \tan^{-1}\frac{y}{x} \end{aligned} \tag{a}$$

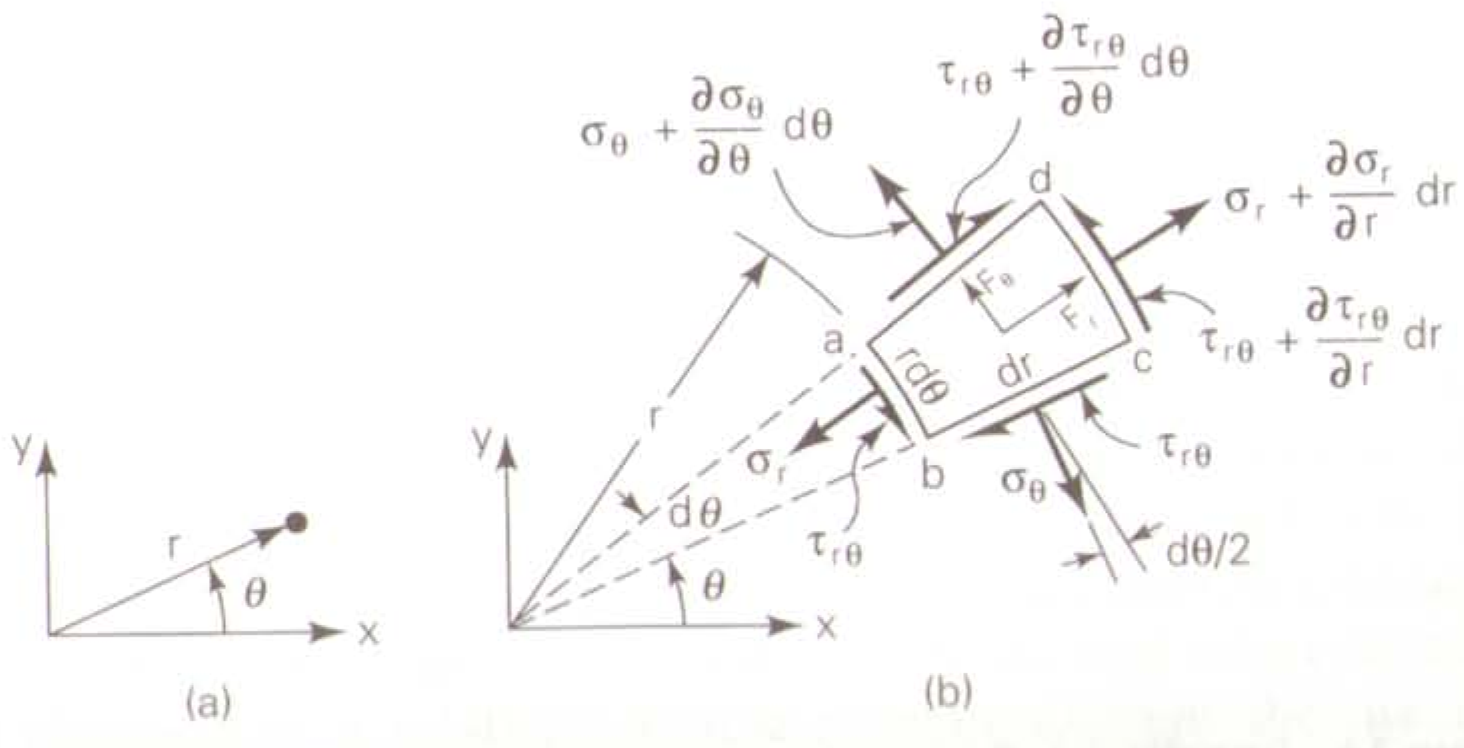

Figure 3.7 (a) Polar coordinates; (b) stress element in polar coordinates.