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MAK 512

Homework #4

5.4. a) Obtain one term solution using Galerkin's method of weighted residuals

$$\frac{d^2y}{dx^2} + y = 2x \qquad 0 \leq x \leq 1$$

$$y(0) = 0$$
$$y(1) = 0$$

$$N_1(x) = x(1-x^2)$$

Using this trial function we get

$$y^*(x) = c_1 x(1-x^2) = c_1 x - c_1 x^3$$

Then $\dfrac{dy^*}{dx} = c_1 - 3c_1 x^2$

$$\frac{d^2y^*}{dx^2} = -6c_1 x$$

We insert y^* and second derivative of y^* into equation to get residual

$$R(x, c_1) = -6c_1 x + c_1 x(1-x^2) - 2x$$

$$R(x, c_1) = (-5c_1 - 2)x - c_1 x^3 \neq 0$$

We will get the coefficient c_1 by the equation

$$\int_0^1 x(1-x^2)\left[(-5c_1-2)x - c_1 x^3\right] dx = 0$$

$$\int_0^1 x^3\left[(5c_1+2)x + c_1 x^3\right] - x\left[(5c_1+2)x + c_1 x^3\right] dx = 0$$

$$\int_0^1 \left[(5c_1+2)x^4 + c_1 x^6 - (5c_1+2)x^2 - c_1 x^4\right] dx = 0$$

$$\int_0^1 [c_1 x^6 + (4c_1+2)x^4 - (5c_1+2)x^2]\,dx = 0$$

$$\left[c_1 \frac{x^7}{7} + (4c_1+2)\frac{x^5}{5} - (5c_1+2)\frac{x^3}{3}\right]\Bigg|_0^1 = 0$$

$$\frac{c_1}{7} + \frac{4c_1+2}{5} - \frac{5c_1+2}{3} = 0$$

$$15c_1 + 84c_1 + 42 - 175c_1 - 70 = 0$$

$$-76c_1 - 28 = 0$$

$$c_1 = -\frac{28}{76} = -\frac{7}{19}$$

Therefore by Galerkin's method, approximate solution is

$$y^*(x) = -\frac{7}{19}x(1-x^2) = \frac{7}{19}(x^3-x)$$

If we solve $y'' + y = 2x$ $\quad y(0)=0$ and $y(1)=0$

Roots of the characteristic equation of the homogeneous part is

$$r_{1,2} = \pm\frac{\sqrt{-4}}{2} = \pm i \Rightarrow y_h(x) = c_1\cos x + c_2 \sin x$$ is the solution of the homogeneous part

Particular solution must have form $y_p(x) = c_3x + c_4$

It must satisfy the differential equation

$$\frac{d^2}{dx^2}(c_3x+c_4) + c_3x + c_4 = 2x$$

$$c_3x + c_4 = 2x \Rightarrow \begin{matrix} c_3 = 2 \\ c_4 = 0 \end{matrix}$$

Therefore solution of the differential equation is $y(x) = y_h(x) + y_p(x)$

$y(x) = c_1 \cos x + c_2 \sin x + 2x$

Using boundary conditions $y(0) = 0 \Rightarrow c_1 = 0$

$$y(1) = c_2\sin(1) + 2 = 0 \Rightarrow c_2 = -\frac{2}{\sin(1)}$$

$$\therefore \quad y(x) = 2x - \frac{2}{\sin(1)}\sin(x)$$

b) $\dfrac{d^2y}{dx^2} + y = 2\sin x \qquad 0 \leq x \leq 1$

$y(0) = 0$

$y(1) = 0$

$N_1(x) = \sin \pi x$

Using the given trial function we get

$y^*(x) = c_1 \sin \pi x$

Then $\dfrac{dy^*}{dx} = c_1 \pi \cos \pi x$

$\dfrac{d^2y^*}{dx^2} = -c_1 \pi^2 \sin \pi x$

To get residual

$R(x, c_1) = -c_1\pi^2 \sin \pi x + c_1 \sin \pi x - 2\sin x \neq 0$

We will get c_1 by the equation

$$\int_0^1 \sin \pi x \left(-c_1\pi^2 \sin \pi x + c_1 \sin \pi x - 2\sin x\right) dx = 0$$

$$\int_0^1 \sin \pi x \left[(c_1 - \pi^2 c_1)\sin \pi x - 2\sin x\right] dx = 0$$ using wolfram alpha webtool

$$\left[\frac{c_1 x}{2} - \frac{1}{2}\pi^2 c_1 x + \frac{1}{4}\pi c_1 \sin(2\pi x) - \frac{c_1 \sin(2\pi x)}{4\pi} - \frac{\sin((\pi-1)x)}{\pi - 1} + \frac{\sin((1+\pi)x)}{1+\pi}\right]\Bigg|_0^1 = 0$$

$$\frac{1}{2}\left(-\pi^2 c_1 + c_1 - \frac{4\pi \sin(1)}{\pi^2 - 1}\right) = 0$$

$$c_1(1 - \pi^2) = \frac{4\pi \sin(1)}{\pi^2 - 1}$$

$$c_1 = \frac{-4\pi \sin(1)}{1 - 2\pi^2 + \pi^4}$$

Therefore by Galerkin's method, approximate solution is

$$y^*(x) = \frac{-4\pi \sin(1)}{1-2\pi^2+\pi^4} \sin(\pi x)$$

If we solve $y'' + y = 2\sin x \quad y(0)=0 \quad y(1)=0$

Roots of the characteristic equation of the homogenous part is

$r_{1,2} = \pm i \Rightarrow y_h(x) = c_1 \cos x + c_2 \sin x$ homogenous solution

Particular solution must have form. $c_3 x \cos x$

It must satisfy the differential equation

$$\frac{d^2}{dx^2}(c_3 x \cos x) + c_3 x \cos x = 2\sin x$$

$$\frac{d}{dx}(c_3 \cos x - c_3 x \sin x) + c_3 x \cos x = -c_3 \sin x - c_3(\sin x + x\cos x) + c_3 x \cos x$$

$$= -c_3 \sin x - c_3 \sin x - c_3 x \cos x + c_3 x \cos x = 2\sin x$$

$$-2c_3 \sin x = 2\sin x$$

$$c_3 = -1$$

Therefore $y_p(x) = -x\cos x$

$y(x) = y_p(x) + y_h(x) = c_1 \cos x + c_2 \sin x - x\cos x$

Using boundary conditions $y(0) = 0 \Rightarrow c_1 = 0$

$y(1) = c_2 \sin(1) - \cos(1) = 0$

$$c_2 = \frac{\cos(1)}{\sin(1)} = \cot(1)$$

$\therefore y(x) = \cot(1)\sin x - x\cos x$

5.7. $N_1(x) = 1 - \frac{x}{L}$ $\qquad N_2(x) = \frac{x}{L}$

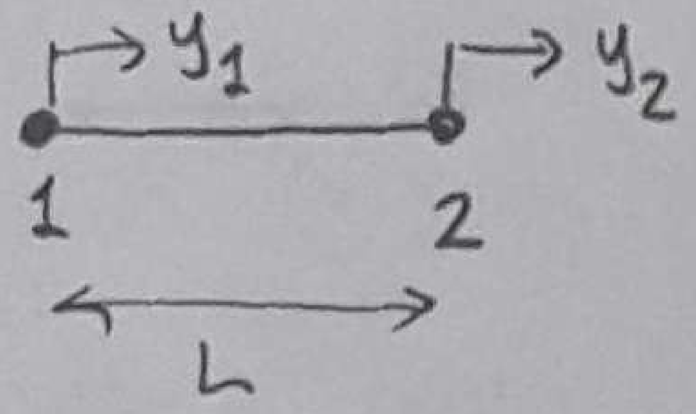
a) One dimensional heat conduction with linearly varying internal heat generation

$$k_x A \frac{d^2 T}{dx^2} + Q_0 A x = 0 \Rightarrow \frac{d^2 T}{dx^2} + \frac{Q_0}{k_x} x = 0$$

For two node element

$$y^{(e)}(x) = N_1(x) y_1 + N_2(x) y_2 \qquad N_1(x) = 1 - \frac{x}{L} \qquad N_2 = \frac{x}{L}$$

y_1 and y_2 are nodal values $\qquad$ we set y as T



We substitute $y^{(e)}$ into equation to get $R^{(e)}$

$$R^{(e)}(x, y_1, y_2) = \frac{d^2 y^{(e)}}{dx^2} + \frac{Q_0}{k_x} x = \frac{d^2}{dx^2}\left[y_1 N_1(x) + y_2 N_2(x)\right] + \frac{Q_0}{k_x} x$$

Applying Galerkin weighted residuals criterion

$$\int_{x_1}^{x_2} N_i(x) R^{(e)}(x, y_1, y_2) dx = \int_{x_1}^{x_2} N_i(x) \left[\frac{d^2}{dx^2} y^{(e)}(x) + \frac{Q_0}{k_x} x\right] dx = 0 \qquad i = 1, 2$$

or $$\int_{x_1}^{x_2} N_i(x) \frac{d^2 y^{(e)}(x)}{dx^2} + \int_{x_1}^{x_2} N_i(x) \frac{Q_0}{k_x} x \, dx = 0 \qquad i = 1, 2$$

Using integration by parts for the first term

$$N_i(x) \frac{dy^{(e)}(x)}{dx} \Bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{dN_i(x)}{dx} \frac{dy^{(e)}(x)}{dx} dx + \int_{x_1}^{x_2} N_i(x) \frac{Q_0}{k_x} x \, dx = 0 \qquad i = 1, 2$$

We will have two seperate equations

$$\int_{x_1}^{x_2} \frac{dN_1(x)}{dx}\frac{dy^{(e)}(x)}{dx}dx = \int_{x_1}^{x_2} N_1(x)\frac{Q_0}{k_x}x\,dx + N_1(x)\frac{dy^{(e)}(x)}{dx}\Bigg|_{x_1}^{x_2}$$

$$\int_{x_1}^{x_2} \frac{dN_2(x)}{dx}\frac{dy^{(e)}(x)}{dx}dx = \int_{x_1}^{x_2} N_2(x)\frac{Q_0}{k_x}x\,dx + N_2(x)\frac{dy^{(e)}(x)}{dx}\Bigg|_{x_1}^{x_2}$$

Then

$$\int_{x_1}^{x_2} \frac{dN_1}{dx}\left[y_1\frac{dN_1}{dx} + y_2\frac{dN_2}{dx}\right]dx = \int_{x_1}^{x_2} N_1(x)\frac{Q_0}{k_x}x\,dx + \frac{dy^{(e)}}{dx}\Bigg|_{x1}$$

$$\int_{x_1}^{x_2} \frac{dN_2}{dx}\left[y_1\frac{dN_1}{dx} + y_2\frac{dN_2}{dx}\right]dx = \int_{x_1}^{x_2} N_2(x)\frac{Q_0}{k_x}x\,dx - \frac{dy^{(e)}}{dx}\Bigg|_{x2}$$

Here if we write in matrix form

$$k_x A \underbrace{\begin{bmatrix} \int_{x_1}^{x_2}\left(\frac{dN_1}{dx}\right)^2 dx & \int_{x_1}^{x_2}\frac{dN_1}{dx}\frac{dN_2}{dx}dx \\ \int_{x_1}^{x_2}\frac{dN_1}{dx}\frac{dN_2}{dx}dx & \int_{x_1}^{x_2}\left(\frac{dN_2}{dx}\right)^2 dx \end{bmatrix}}_{\text{stiffness matrix}} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \underbrace{Q_0 A \begin{Bmatrix} \int_{x_1}^{x_2} N_1(x)\frac{Q_0}{k_x}x\,dx \\ \int_{x_1}^{x_2} N_2(x)\frac{Q_0}{k_x}x\,dx \end{Bmatrix}}_{\text{forcing matrix}} + k_x A \begin{Bmatrix} \frac{dy^{(e)}}{dx}\Big|_{x_1} \\ \frac{dy^{(e)}}{dx}\Big|_{x_2} \end{Bmatrix}$$

$\frac{dN_1}{dx} = -\frac{1}{L}$ $\quad \frac{dN_2}{dx} = \frac{1}{L}$ Therefore stiffness matrix will be

$$\left[k^{(e)}\right] = k_x A \begin{bmatrix} \int_{x_1}^{x_2}\frac{1}{L^2}dx & \int_{x_1}^{x_2}-\frac{1}{L^2}dx \\ \int_{x_1}^{x_2}-\frac{1}{L^2}dx & \int_{x_1}^{x_2}\frac{1}{L^2}dx \end{bmatrix} = \frac{k_x A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

since $x_2 - x_1 = L$

Forcing vector

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = Q_0 A \begin{Bmatrix} \int_{x_1}^{x_2} x\left(1-\frac{x}{L}\right)dx \\ \int_{x_1}^{x_2} x\left(\frac{x}{L}\right)dx \end{Bmatrix} = Q_0 A \begin{Bmatrix} \frac{L^2}{2} - \frac{L^3}{3L} \\ \frac{L^3}{3L} \end{Bmatrix} = Q_0 A L^2 \begin{Bmatrix} \frac{1}{6} \\ \frac{1}{3} \end{Bmatrix}$$

b) One dimensional heat conduction with surface convection

$$k_x A \frac{d^2T}{dx^2} - hPT = hPT_a \Rightarrow \frac{d^2T}{dx^2} = \frac{hP}{k_x A}(T_a - T)$$

$$\frac{d^2T}{dx^2} + \frac{hP}{k_x A} T - \frac{hP}{k_x A} T_a = 0$$ we set y as T

$$y^{(e)} = N_1(x) y_1 + N_2(x) y_2 \qquad N_1(x) = 1 - \frac{x}{L} \qquad N_2 = \frac{x}{L}$$

$R^{(e)}$ will be

$$R^{(e)}(x, y_1, y_2) = \frac{d^2 y^{(e)}}{dx^2} + \frac{hP}{k_x A} y^{(e)} - \frac{hPT_a}{k_x A} \neq 0$$

Applying Galerkin weighted residuals criterion

$$\int_{x_1}^{x_2} N_i(x) R^{(e)}(x, y_1, y_2)\, dx = \int_{x_1}^{x_2} N_i(x) \left[\frac{d^2 y^{(e)}}{dx^2} + \frac{hP}{k_x A} y^{(e)} - \frac{hPT_a}{k_x A} \right] dx = 0 \quad i = 1, 2$$

or $$\int_{x_1}^{x_2} N_i(x) \frac{d^2 y^{(e)}}{dx^2} dx + \int_{x_1}^{x_2} \left[\frac{hP}{k_x A} y^{(e)} - \frac{hPT_a}{k_x A} \right] dx = 0 \quad i = 1, 2$$

Using integration by parts for first term

$$N_i(x) \frac{dy^{(e)}(x)}{dx} \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{dN_i(x)}{dx} \frac{dy^{(e)}(x)}{dx} dx + \frac{hP}{k_x A} \int_{x_1}^{x_2} (y^{(e)} - T_a) dx = 0 \quad i = 1, 2$$

we will have two equations

$$\int_{x_1}^{x_2} \frac{dN_1}{dx} \frac{dy^{(e)}(x)}{dx} dx = \frac{hP}{k_x A} \int_{x_1}^{x_2} (y^{(e)} - T_a) dx + \frac{dy^{(e)}}{dx} \Big|_{x_1}$$

$$\int_{x_1}^{x_2} \frac{dN_2}{dx} \frac{dy^{(e)}(x)}{dx} dx = \frac{hP}{k_x A} \int_{x_1}^{x_2} (y^{(e)} - T_a) dx - \frac{dy^{(e)}}{dx} \Big|_{x_2}$$

Note that

$$\frac{dy^{(e)}}{dx} = y_1 \frac{dN_1}{dx} + y_2 \frac{dN_2}{dx} \qquad y^{(e)} = y_1 N_1 + y_2 N_2$$

Here we have the equations in matrix form

$$\frac{dN_1}{dx} = -\frac{1}{L} \qquad \frac{dN_2}{dx} = \frac{1}{L} \qquad \text{Therefore}$$

$$\int_{x_1}^{x_2} \frac{dN_1}{dx}\frac{dy^{(e)}}{dx}dx - \frac{hP}{k_xA}\int_{x_1}^{x_2}(y^{(e)}-T_a)\,dx = \left.\frac{dy^{(e)}}{dx}\right|_{x_1}$$

$$\int_{x_1}^{x_2} \frac{dN_2}{dx}\frac{dy^{(e)}}{dx}dx - \frac{hP}{k_x}\int_{x_1}^{x_2}(y^{(e)}-T_a)\,dx = -\left.\frac{dy^{(e)}}{dx}\right|_{x_2}$$

$$\int_{x_1}^{x_2} \frac{dN_1}{dx}\left[y_1\frac{dN_1}{dx}+y_2\frac{dN_2}{dx}\right]dx - \frac{hP}{k_xA}\int_{x_1}^{x_2} y_1N_1+y_2N_2\,dx = \frac{hP}{k_xA}\int_{x_1}^{x_2} T_a\,dx + \left.\frac{dy^{(e)}}{dx}\right|_{x_1}$$

$$\int_{x_1}^{x_2} \frac{dN_2}{dx}\left[y_1\frac{dN_1}{dx}+y_2\frac{dN_2}{dx}\right]dx - \frac{hP}{k_xA}\int_{x_1}^{x_2} y_1N_1+y_2N_2\,dx = \frac{hP}{k_xA}\int_{x_1}^{x_2} T_a\,dx - \left.\frac{dy^{(e)}}{dx}\right|_{x_2}$$

Stiffness matrix will be

$$[k^{(e)}] = \begin{bmatrix} \int_{x_1}^{x_2}\left[\left(\frac{dN_1}{dx}\right)^2 - \frac{hP}{k_xA}N_1\right]dx & \int_{x_1}^{x_2}\left[\left(\frac{dN_1}{dx}\frac{dN_2}{dx}\right) - \frac{hP}{k_xA}N_2\right]dx \\ \int_{x_1}^{x_2}\left[\left(\frac{dN_1}{dx}\frac{dN_2}{dx}\right) - \frac{hP}{k_xA}N_1\right] & \int_{x_1}^{x_2}\left[\left(\frac{dN_2}{dx}\right)^2 - \frac{hP}{k_xA}N_2\right]dx \end{bmatrix}$$

$$[k^{(e)}] = \begin{bmatrix} \left.(+\frac{1}{L^2}) - \left(\frac{hP}{k_xA}\left(x-\frac{x^2}{2L}\right)\right)\right|_{x_1}^{x_2} & \left.(-\frac{1}{L^2}) - \left(\frac{hP}{k_xA}\frac{x^2}{2L}\right)\right|_{x_1}^{x_2} \\ \left.(-\frac{1}{L^2}) - \left(\frac{hP}{k_xA}\left(x-\frac{x^2}{2L}\right)\right)\right|_{x_1}^{x_2} & \left.(+\frac{1}{L^2}) - \left(\frac{hP}{k_xA}\frac{x^2}{2L}\right)\right|_{x_1}^{x_2} \end{bmatrix}$$

$$[k^{(e)}] = \frac{1}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \frac{hP}{k_xA}\begin{bmatrix} \left.\left(x-\frac{x^2}{2L}\right)\right|_{x_1}^{x_2} & \left.\left(\frac{x^2}{2L}\right)\right|_{x_1}^{x_2} \\ \left.\left(x-\frac{x^2}{2L}\right)\right|_{x_1}^{x_2} & \left.\left(\frac{x^2}{2L}\right)\right|_{x_1}^{x_2} \end{bmatrix}$$

Forcing vector will be

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{hP}{k_xA}\int_{x_1}^{x_2} T_a\,dx \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{hPT_aL}{k_xA}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

c) Torsion of an elastic circular cylinder

$$JG\frac{d^2\theta}{dx^2}=0 \qquad \text{we set } y \text{ as } \theta$$

$$y^{(e)}=N_1(x)y_1+N_2(x)y_2 \qquad N_1(x)=1-\frac{x}{L} \qquad N_2=\frac{x}{L}$$

$R^{(e)}$ will be

$$R^{(e)}(x,y_1,y_2)=JG\frac{d^2y^{(e)}}{dx^2}\neq 0$$

Applying Galerkin weighted residuals criteria

$$\int_{x_1}^{x_2}N_i(x)R^{(e)}(x,y_1,y_2)\,dx=\int_{x_1}^{x_2}N_i(x)\left[JG\frac{d^2y^{(e)}}{dx^2}\right]dx=0 \quad i=1,2$$

Using integration by parts

$$JG\,N_i(x)\frac{dy^{(e)}}{dx}(x)\bigg|_{x_1}^{x_2}-\int_{x_1}^{x_2}JG\frac{d\,N_i(x)}{dx}\frac{dy^{(e)}(x)}{dx}\,dx=0 \quad i=1,2$$

We will have two equations

$$JG\int_{x_1}^{x_2}\frac{dN_1}{dx}\frac{dy^{(e)}}{dx}dx=JG\frac{dy^{(e)}}{dx}\bigg|_{x_1}$$

$$JG\int_{x_1}^{x_2}\frac{dN_2}{dx}\frac{dy^{(e)}}{dx}dx=-JG\frac{dy^{(e)}}{dx}\bigg|_{x_2}$$

$$\Rightarrow \int_{x_1}^{x_2}\frac{dN_1}{dx}\left[y_1\frac{dN_1}{dx}+y_2\frac{dN_2}{dx}\right]dx=\frac{dy^{(e)}}{dx}\bigg|_{x_1}$$

$$\int_{x_1}^{x_2}\frac{dN_2}{dx}\left[y_1\frac{dN_1}{dx}+y_2\frac{dN_2}{dx}\right]dx=-\frac{dy^{(e)}}{dx}\bigg|_{x_2}$$

Note that $\frac{dy^{(e)}}{dx}=y_1\frac{dN_1}{dx}+y_2\frac{dN_2}{dx}$ $\qquad y^{(e)}=y_1N_1+y_2N_2$

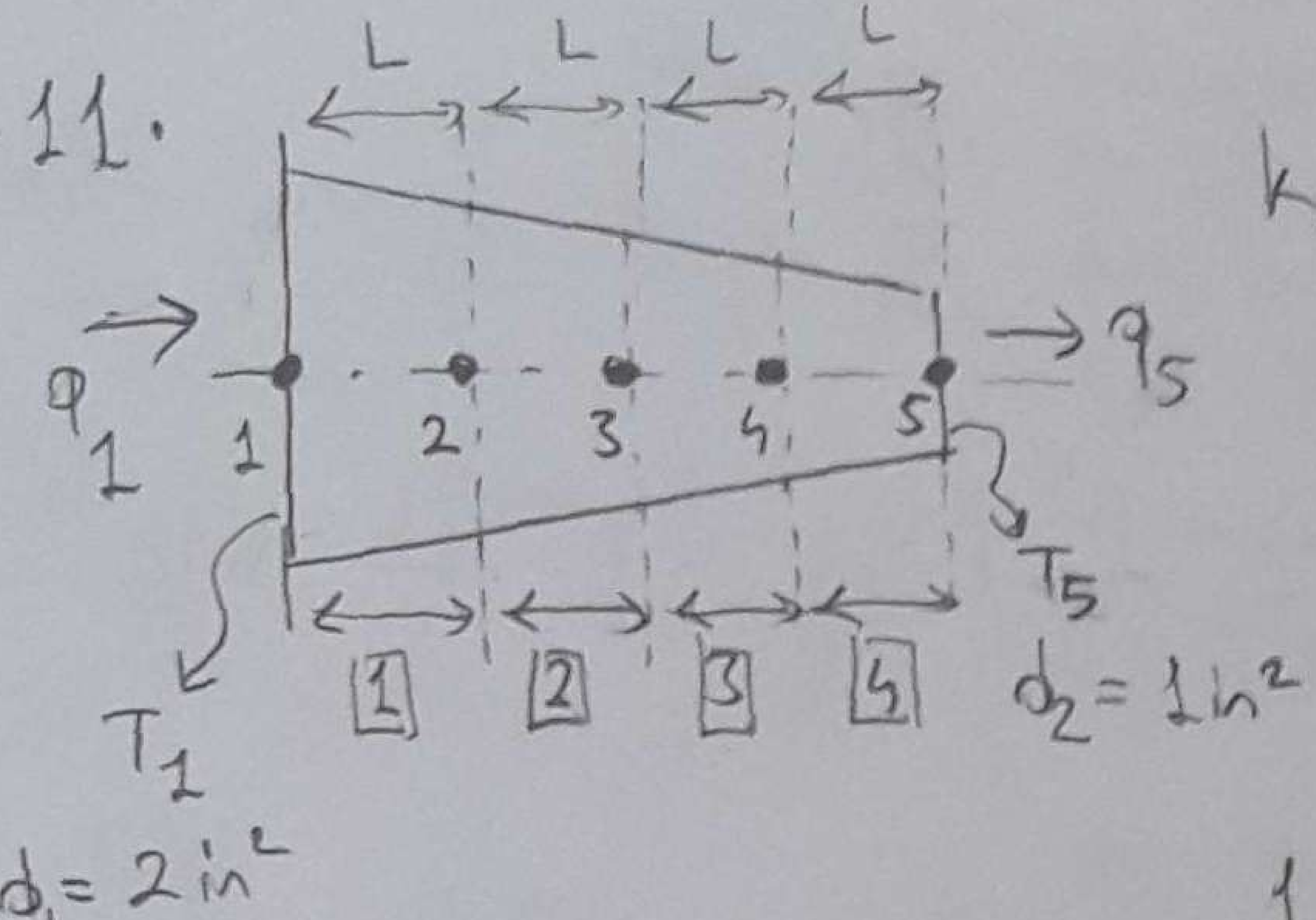
Therefore in matrix form

$$[k_e]=JG\begin{bmatrix}\int_{x_1}^{x_2}\left(\frac{dN_1}{dx}\right)^2dx & \int_{x_1}^{x_2}\left(\frac{dN_1}{dx}\right)\left(\frac{dN_2}{dx}\right)dx\\ \int_{x_1}^{x_2}\left(\frac{dN_1}{dx}\right)\left(\frac{dN_2}{dx}\right)dx & \int_{x_1}^{x_2}\left(\frac{dN_2}{dx}\right)^2dx\end{bmatrix}=\frac{JG}{L}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}$$

Forcing vector

Note that $\frac{dN_1}{dx}=-\frac{1}{L}$ $\quad \frac{dN_2}{dx}=\frac{1}{L}$ $\quad x_2-x_1=L$ $\qquad F=\begin{bmatrix}0\\0\end{bmatrix}$

S.11.



$k_x = 64 \dfrac{btu}{hr\text{-}ft\text{-}°F}$ $L = 1$ inch

$T_1 = 212\,°F$

$T_5 = 80\,°F$

Perfecty insulated on periphery

1D conduction heat transfer problem

$$k_x A \frac{d^2T}{dx^2} = 0$$

Conductance matrix for an element is

$$[k^{(e)}] = \frac{k_x A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

We have element-node connectivity table as shown

	Element			
Node	1	2	3	4
1	1			
2	2	1		
3		2	1	
4			2	1
5				2

The global stiffness matrix is

$$\frac{k_x}{L}\begin{bmatrix} A_1 & -A_1 & 0 & 0 & 0 \\ -A_1 & A_1+A_2 & -A_2 & 0 & 0 \\ 0 & -A_2 & A_2+A_3 & -A_3 & 0 \\ 0 & 0 & -A_3 & A_3+A_4 & -A_4 \\ 0 & 0 & 0 & -A_4 & A_4 \end{bmatrix}$$

A is linearly varying we can compute A_1, A_2, A_3, A_4

Since we have no heat generation and no convection and no radiation $q_1 = q_5$

If we write the thermal equation in matrix form

$$\frac{k_x}{L}\begin{bmatrix} A_1 & -A_1 & 0 & 0 & 0 \\ -A_1 & A_1+A_2 & -A_2 & 0 & 0 \\ 0 & -A_2 & A_2+A_3 & -A_3 & 0 \\ 0 & 0 & -A_3 & A_3+A_4 & -A_4 \\ 0 & 0 & 0 & -A_4 & A_4 \end{bmatrix}\begin{bmatrix} 212 \\ T_2 \\ T_3 \\ T_4 \\ 80 \end{bmatrix} = \begin{bmatrix} q_1 A_1 \\ 0 \\ 0 \\ 0 \\ q_5 A_5 \end{bmatrix}$$

Since T_1 and T_5 are fixed, we need to solve the matrix equation above.

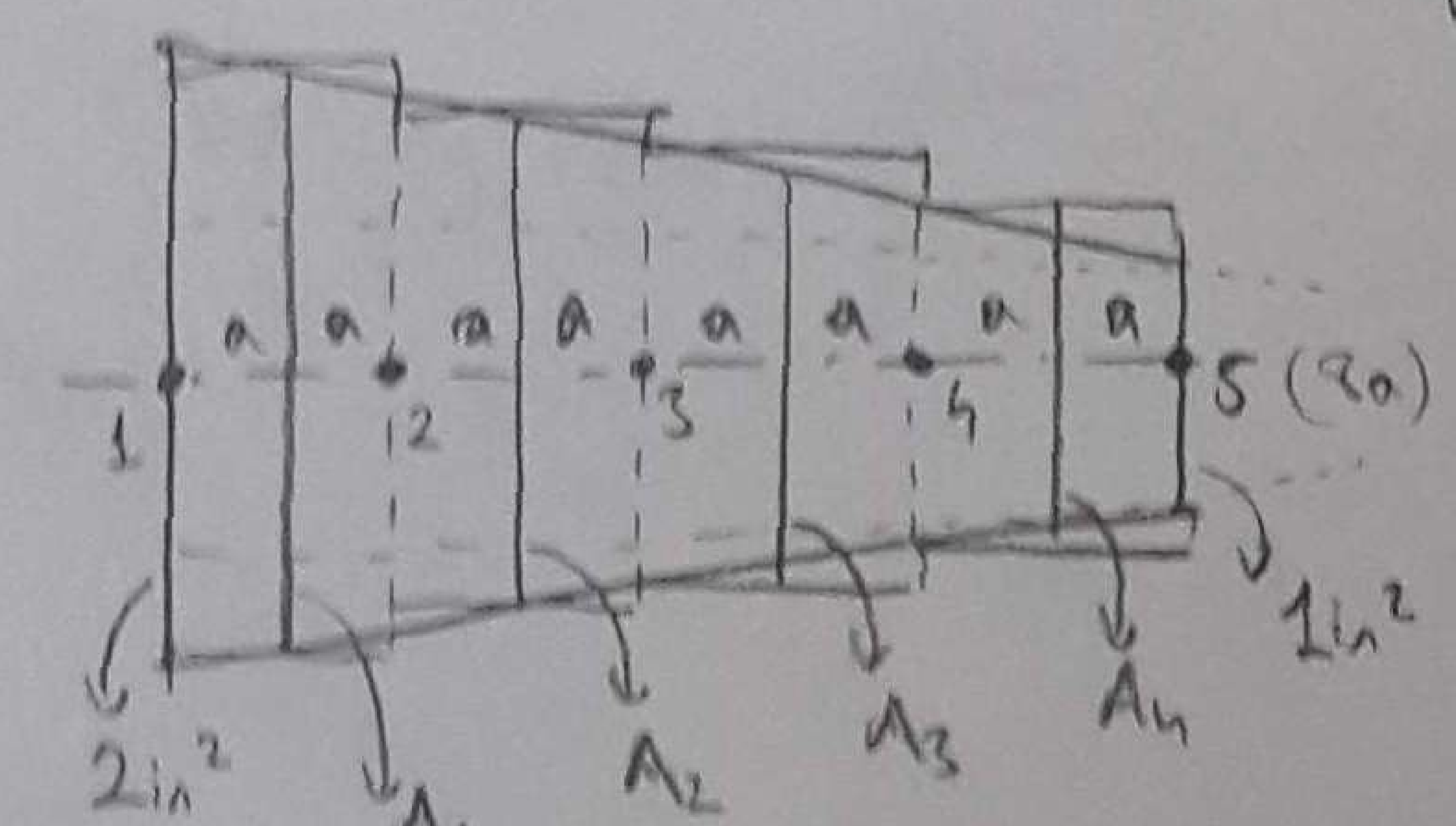
We have 5 equations and 5 unknowns.

We must satisfy $q_1 A_1 = -q_5 A_5 = Q$ since there is no heat generation

We can compute A_1, A_2, A_3, A_4 as follows

Let length of each element be $2a$

We compute areas at the middle of each element



$$\frac{A_4}{2} = \frac{(9a)^2}{(16a)^2} \Rightarrow A_4 = 2\left(\frac{9}{16}\right)^2$$

$$\frac{A_3}{2} = \frac{(11a)^2}{(16a)^2} \Rightarrow A_3 = 2\left(\frac{11}{16}\right)^2$$

$$\frac{A_2}{2} = \frac{(13a)^2}{(16a)^2} \Rightarrow A_2 = 2\left(\frac{13}{16}\right)^2$$

$$\frac{A_1}{2} = \frac{(15a)^2}{(16a)^2} \Rightarrow A_1 = 2\left(\frac{15}{16}\right)^2$$

We use calculator for the matrix operations

$$k_x = 64 \frac{btu}{hr \cdot ft \cdot {}^\circ F} = 64 \cdot \frac{9331.8\,[in][lbf]}{3600\,[s] \cdot 12\,[in]\,{}^\circ F} = 13.825 \frac{[in][lbf]}{[s][in][{}^\circ F]}$$

$$L = 1in \Rightarrow \frac{k_x}{L} = 13.825 \frac{[lbf]}{[s][in][{}^\circ F]}$$

Let's use 2nd and 4th equations and we get

$$\frac{k_x}{L}\begin{bmatrix} A_1+A_2 & -A_2 & 0 \\ -A_2 & A_2+A_3 & -A_3 \\ 0 & -A_3 & A_3+A_4 \end{bmatrix}\begin{bmatrix} T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 212\, \frac{k_x A_1}{L} \\ 0 \\ 80\, \frac{k_x A_4}{L} \end{bmatrix}$$

From here

$$T_2 = 193.06\ °F$$
$$T_3 = 167.84\ °F$$
$$T_4 = 132.61\ °F$$

The heat transfer at boundaries will be

$$\left.\begin{aligned} q_1 A_1 &= Q = 460.3\ [in][lbf] \\ q_5 A_5 &= -Q = -460.3\ [in][lbf] \end{aligned}\right\}$$ Amount of heat transfer at the boundary nodes.

5.15 $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$

Laplace's equation

To apply Galerkin's method we will need weighting functions

We must assume

$$\phi(x,y) = (N_1 x_1 + N_2 x_2)(M_1 y_1 + M_2 y_2)$$

Then residuals will be

$$R^{(e)}(x, y, x_1, x_2, y_1, y_2) = \nabla^2\left[(N_1 x_1 + N_2 x_2)(M_1 y_1 + M_2 y_2)\right]$$

Finally we must apply Galerkin weighted residuals criteria

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} M_j(y) N_i(x)\, R^{(e)}(x, y, x_1, x_2, y_1, y_2)\, dx\, dy = 0 \qquad \begin{array}{l} i = 1, 2 \\ j = 1, 2 \end{array}$$