TOBB EKONOMİ VE TEKNOLOJİ ÜNİVERSİTESİ

Makina Mühendisliği Bölümü

MAK506 ELASTİSİTE TEORİSİ

Bahar Dönemi 2008-2009

Ara Sınav 2

Dr. Mehmet Ali Güler

Ad, Soyad ______________________

21 Mart 2009 Cumartesi

Öğrenci No ______________________

Verilen Zaman: **2,5 saat (10:00-12:30)**

Kitap ve Notlar Kapalı

- *Her soruyu dikkatle okuyun.*
- *Yaptığınız işlemleri gösterin.*
- *Sonuçları verilen kutular içine yazın.*
- *Temiz çalışın.*
- *Sınav salonunda cep telefonu kullanmak yasaktır.*

Soru No	Maksimum Puan	Puan
1	20	
2	25	
3	25	
4	25	
5	25	
Toplam	**120**	

Ön sayfa dahil, bu sınav kağıdında toplam **(6)** sayfa vardır.

1. **(20 puan)** (Pr. 2.13, Advanced Strength and Applied Stress Analysis by Budynas) The state of stress at a point within a structure relative to an *xyz* coordinate system is given by,

$$[\sigma]=\begin{bmatrix}20 & 35 & 25\\ 35 & 0 & -15\\ 25 & -15 & 10\end{bmatrix}\text{MPa}$$

If the coordinate location of the point in space is $(1,1,-2)$ determine the normal and shear stresses at the point and on an internal surface established by a sphere with equation

$$x^2+(y-2)^2+z^2=6$$

$$\vec{N}=-\nabla\left(x^2+(y-2)^2+z^2-6\right)$$

$$\vec{N}=-2x\hat{i}-2(y-2)\hat{j}-2z\hat{k}$$

@ $(1,1,-2)$

$$\vec{N}=-2\hat{i}+2\hat{j}+4\hat{k}$$

$$\hat{n}=\frac{1}{\sqrt{6}}\left(-\hat{i}+\hat{j}+2\hat{k}\right)$$

$$\vec{T}^n=\sigma_{ij}\cdot n_j=\begin{bmatrix}20 & 35 & 25\\ 35 & 0 & -15\\ 25 & -15 & 10\end{bmatrix}\begin{Bmatrix}-1/\sqrt{6}\\ 1/\sqrt{6}\\ 2/\sqrt{6}\end{Bmatrix}=\begin{Bmatrix}\frac{65}{\sqrt{6}}\\ -\frac{65}{\sqrt{6}}\\ -\frac{20}{\sqrt{6}}\end{Bmatrix}$$

$$\vec{T}^n=\frac{1}{\sqrt{6}}\left(65\hat{i}-65\hat{j}-20\hat{k}\right)$$

$$|\vec{T}^n|=\frac{1}{\sqrt{6}}\sqrt{65^2+(-65)^2+(-20)^2}=38.405\text{ MPa}$$

$$\sigma_{nn}=\vec{T}^n\cdot\hat{n}=\frac{1}{\sqrt{6}}\left(65\hat{i}-65\hat{j}-20\hat{k}\right)\cdot\frac{1}{\sqrt{6}}\left(-\hat{i}+\hat{j}+2\hat{k}\right)$$

$$=\frac{1}{6}(-65-65-40)=-\frac{170}{6}=-28.33\text{ MPa}$$

$$\sigma_{nt}=\sqrt{|\vec{T}^n|^2-\sigma_{nn}^2}=\sqrt{(38.405)^2-(-28.33)^2}$$

$$\sigma_{nt}=25.926\text{ MPa}$$

2. **(25 points)** (Pr. 4.7, Elasticity, M. H. Sadd) The displacements in an elastic material are given by

$$u = -\frac{M(1-\nu^2)}{EI}xy, \qquad v = \frac{M(1+\nu)\nu}{2EI}y^2 + \frac{M(1-\nu^2)}{2EI}\left(x^2 - \frac{l^2}{4}\right), \qquad w = 0,$$

where M, E, I, and l are constant parameters. Determine the corresponding strain and stress fields and show that this problem represents the pure bending of a rectangular beam in the x,y plane.

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right] \quad \text{———— (1)}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0 \quad \text{———— (2)}$$

From (1) and (2)

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \quad \text{———— (3)}$$

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] \quad \text{———— (4)}$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right] \quad \text{———— (5)}$$

Substituting (3) into (4) & (5) we have

$$\varepsilon_{xx} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right] \quad \text{———— (6)}$$

$$\varepsilon_{yy} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_{yy} - \nu\sigma_{xx}\right] \quad \text{———— (7)}$$

From (6) & (7) we can solve for σ_{xx} & σ_{yy}

$$\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy}\right] \quad \text{———— (8)}$$

$$\sigma_{yy} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{yy} + \nu\varepsilon_{xx}\right] \quad \text{———— (9)}$$

Also

$$\sigma_{xy} = \frac{1}{2\mu}\varepsilon_{xy} = \frac{1+\nu}{E}\varepsilon_{xy} \quad \text{———— (10)}$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -\frac{M(1-\nu^2)}{EI}\, y \qquad (11)$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \frac{M(1+\nu)\nu}{EI}\, y \qquad (12)$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

$$\varepsilon_{xy} = \frac{1}{2}\left[-\frac{M(1-\nu^2)}{EI}\, x + \frac{M(1-\nu^2)}{EI}\, x \right]$$

$$\varepsilon_{xy} = 0 \qquad (13)$$

Substituting (11) and (12) into (9)

$$\sigma_{yy} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{M(1+\nu)\nu}{EI}\, y + \nu\left(-\frac{M(1-\nu^2)}{EI}\, y \right) \right]$$

$$\sigma_{yy} = 0 \qquad (14)$$

Substituting (11) and (12) into (8)

$$\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\left(-\frac{M(1-\nu^2)}{EI}\, y \right) + \nu\, \frac{M(1+\nu)\nu}{EI}\, y \right]$$

$$= \frac{E}{\cancel{(1+\nu)}(1-2\nu)} \cdot \frac{\cancel{(1+\nu)}\, M y}{EI}\left[-(1-\nu)^2 + \nu^2 \right]$$

$$= -\frac{\cancel{E}}{\cancel{(1-2\nu)}} \cdot \frac{M y}{\cancel{E} I}\, \cancel{(1-2\nu)}$$

$$\sigma_{xx} = -\frac{M y}{I} \qquad (15)$$

Pure bending of a rectangular beam in the x,y plane

3. **(25 points)** (Pr. 3.7, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase) With respect to $Ox_1x_2x_3$ the stress state is given in terms of the coordinates by the matrix

$$\left[\sigma_{ij}\right] = \begin{bmatrix} x_1x_2 & x_2^2 & 0 \\ x_2^2 & x_2x_3 & x_3^2 \\ 0 & x_3^2 & x_3x_1 \end{bmatrix}$$

Determine

(a) The body force components as functions of the coordinates if the equilibrium equations are to be satisfied everywhere

(b) The stress vector at point P(1,2,3) on the plane whose outward unit normal makes equal angles with the positive coordinate axes.

(a) $\sigma_{ij,j} + F_i = 0$

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + F_1 = 0 \quad \text{———— (1)}$$

$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + F_2 = 0 \quad \text{———— (2)}$$

$$\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + F_3 = 0 \quad \text{———— (3)}$$

From (1)

$$x_2 + 2x_2 + F_1 = 0 \quad \Rightarrow \quad F_1 = -3x_2$$

From (2)

$$x_3 + 2x_3 + F_2 = 0 \quad \Rightarrow \quad F_2 = -3x_3$$

From (3)

$$x_1 + F_3 = 0 \quad \Rightarrow \quad F_3 = -x_1$$

(b) stress vector at point $P(1,2,3)$

$$[\sigma_{ij}] = \begin{bmatrix} (1)(2) & 2^2 & 0 \\ 2^2 & (2)(3) & (3)^2 \\ 0 & (3)^2 & (3)(1) \end{bmatrix} = \begin{bmatrix} 2 & 4 & 0 \\ 4 & 6 & 9 \\ 0 & 9 & 3 \end{bmatrix}$$

$$\hat{n} = \frac{1}{\sqrt{3}}\left(\hat{i} + \hat{j} + \hat{k}\right)$$

$$\vec{T}_i^{\,n} = \sigma_{ij} \cdot n_j$$

$$= \begin{bmatrix} 2 & 4 & 0 \\ 4 & 6 & 9 \\ 0 & 9 & 3 \end{bmatrix} \begin{Bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{Bmatrix} = \begin{Bmatrix} 6/\sqrt{3} \\ 19/\sqrt{3} \\ 12/\sqrt{3} \end{Bmatrix}$$

$$\vec{T}_i^{\,n} = \frac{1}{\sqrt{3}}\left(6\hat{i} + 19\hat{j} + 12\hat{k}\right)$$

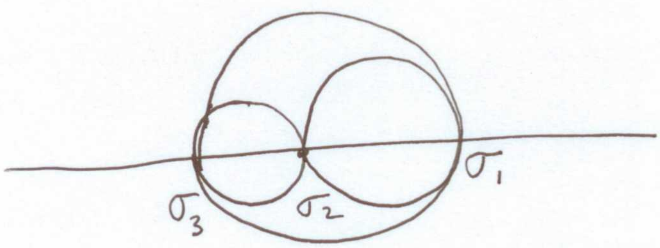
4. **(25 Mase)** (Pr. 3.16, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase) At point P, the stress matrix relative to axes $Px_1x_2x_3$ is given in MPa by

$$[\sigma_{ij}] = \begin{bmatrix} 5 & a & -a \\ a & 0 & b \\ -a & b & 0 \end{bmatrix}$$

Where a and b are unspecified. At the same point relative to axes $Px_1^*x_2^*x_3^*$ the matrix is

$$[\sigma_{ij}^*] = \begin{bmatrix} \sigma_{\mathrm{I}} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sigma_{\mathrm{III}} \end{bmatrix}$$

If the magnitude of the maximum shear stress at P is 5.5 Mpa, determine σ_{I} and σ_{III}



$$\boxed{\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = 5.5 \text{ MPa}}$$

$\therefore$

$$\sigma_1 - \sigma_3 = 11 \text{ MPa} \quad \text{———} (1)$$

we know that $\sigma_2 = 2$ MPa

Principal stresses may be found by

$$\begin{vmatrix} 5-\lambda & a & -a \\ a & 0-\lambda & b \\ -a & b & 0-\lambda \end{vmatrix} = (5-\lambda)(\lambda^2 - b^2) - a(-a\lambda + ab) - a(ab - \lambda a)$$

$$= 5\lambda^2 - 5b^2 - \lambda^3 + \lambda b^2 + a^2\lambda - a^2 b - a^2 b + a^2\lambda$$

$$= -\lambda^3 + 5\lambda^2 + (b^2 + 2a^2)\lambda - 5b^2 - 2a^2 b = 0$$

characteristic equation is

$$-\lambda^3 + 5\lambda^2 + (b^2 + 2a^2)\lambda - 5b^2 - 2a^2 b = 0 \quad \text{———} (2)$$

since we know one of the roots of this equation we can write

$$(\lambda-2)(\lambda-\sigma_1)(\lambda-\sigma_3) = (\lambda-2)\left[\lambda^2-(\sigma_3+\sigma_1)\lambda+\sigma_1\sigma_3\right]$$

$$= \lambda^3-(\sigma_3+\sigma_1)\lambda^2+\sigma_1\sigma_3\lambda$$
$$-2\lambda^2+2(\sigma_3+\sigma_1)\lambda-2\sigma_1\sigma_3$$

$$= \lambda^3-(2+\sigma_3+\sigma_1)\lambda^2+\left[\sigma_1\sigma_3+2(\sigma_3+\sigma_1)\right]\lambda$$
$$-2\sigma_1\sigma_3$$

or multibly by -1

characteristic equation becomes

$$-\lambda^3+(2+\sigma_3+\sigma_1)\lambda^2-\left[\sigma_1\sigma_3+2(\sigma_3+\sigma_1)\right]\lambda+2\sigma_1\sigma_3=0 \quad \text{— (3)}$$

equating (2) and (3) we have

$$2+\sigma_3+\sigma_1 = 5$$

$$\therefore \quad \sigma_1+\sigma_3 = 3 \quad \text{——— (4)}$$

using (4) and (1)

$$\sigma_1-\sigma_3 = 11$$
$$+\quad \sigma_1+\sigma_3 = 3$$
$$\overline{\quad 2\sigma_1 = 14 \quad} \Rightarrow \quad \sigma_1 = 7 \text{ MPa}$$

using (4)

$$7+\sigma_3 = 3 \quad \Rightarrow \quad \sigma_3 = -4 \text{ MPa}$$

5. **(25 puan)** (Pr. 3.22, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase) The state of stress referred to axes $Px_1x_2x_3$ is given in MPa by the matrix

$$\left[\sigma_{ij}\right]=\begin{bmatrix} 9 & 12 & 0 \\ 12 & -9 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Determine

(a) The normal and shear components, σ_N and σ_S, respectively, on the plane at P whose unit normal is

$$\hat{n}=\frac{1}{5}\left(4\hat{e}_1+3\hat{e}_2\right)$$

(b) Verify the results determined in (a) by a Mohr's circle construction.

$$n = \frac{4}{5}\hat{e}_1 + \frac{3}{5}\hat{e}_2$$

$$\vec{T}_i^n = \sigma_{ij}\cdot n_j = \begin{bmatrix} 9 & 12 & 0 \\ 12 & -9 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{Bmatrix} 4/5 \\ 3/5 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 72/5 \\ 21/5 \\ 0 \end{Bmatrix}$$

$$\vec{T}^n = \frac{72}{5}\hat{e}_1 + \frac{21}{5}\hat{e}_2$$

$$|\vec{T}^n| = \sqrt{\left(\frac{72}{5}\right)^2+\left(\frac{21}{5}\right)^2} = 15 \text{ MPa}$$

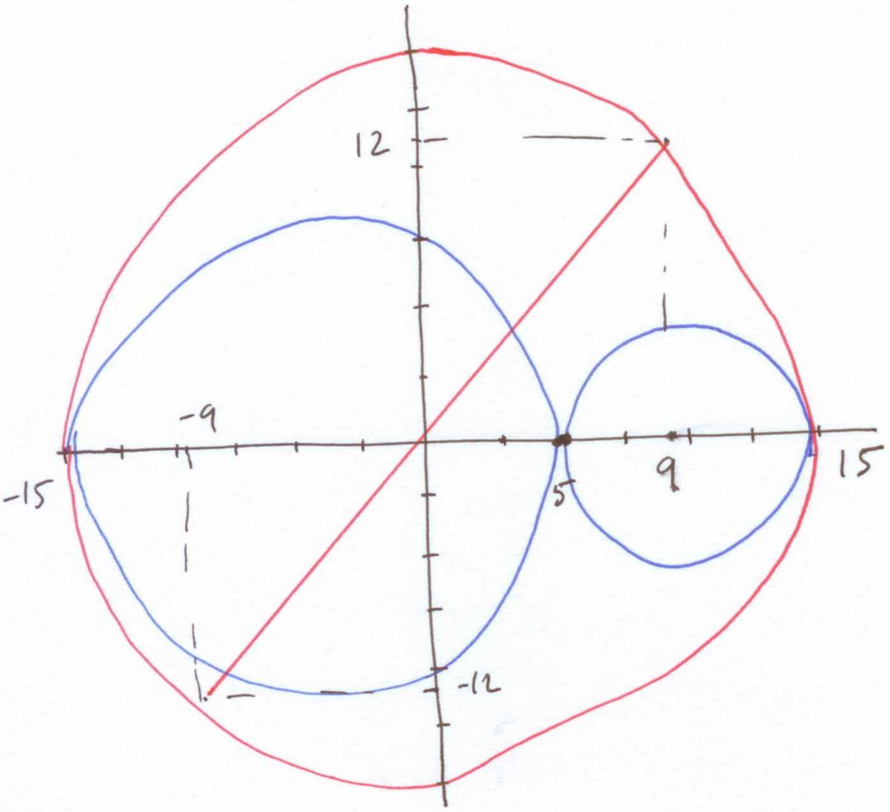
$$\sigma_N = \vec{T}^n\cdot\hat{n} = \left(\frac{72}{5}\hat{e}_1 + \frac{21}{5}\hat{e}_2\right)\cdot\left(\frac{4}{5}\hat{e}_1 + \frac{3}{5}\hat{e}_2\right)$$

$$= \frac{288}{25} + \frac{63}{25} = 14.04 \text{ MPa}$$

$$|\vec{T}^n|^2 = \sigma_N^2 + \sigma_S^2$$

$$\sqrt{\sigma_S^2} = \sqrt{|\vec{T}^n|^2 - \sigma_N^2}$$

$$\sigma_S = 5.28 \text{ MPa}$$



$\sigma_1 = 15$ MPa

$\sigma_2 = 5$ MPa

$\sigma_3 = -15$ MPa