1. (35 puan) Aşağıda verilen ikinci ve birinci dereceden tensörler için

$$a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \qquad b_i = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

aşağıdaki değerleri hesaplayınız:

(a) $c_{ij} = \frac{1}{2}\left(a_{ij} + a_{ji}\right)$, $(i, j = 1,2,3)$ (7 puan)

(b) $d_{ij} = \frac{1}{2}\left(a_{ij} - a_{ji}\right)$, $(i, j = 1,2,3)$ (7 puan)

(c) $a_{ij}a_{kj}$, $(i, j, k = 1,2,3)$ (7 puan)

(d) $a_{ij}b_j$, $(i, j = 1,2,3)$ (7 puan)

(e) $c_{ij}d_{ij}$, $(i, j = 1,2,3)$ (7 puan)

Not: c_{ij} (a) şıkkından, d_{ij} (b) şıkkından hesaplanacaktır.

(a) $$c_{ij} = \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 4 & 1.5 \\ 0.5 & 1.5 & 1 \end{bmatrix}$$ second order tensor

(b) $$d_{ij} = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.5 & 0.5 \\ -0.5 & 0 & 0.5 \\ -0.5 & -0.5 & 0 \end{bmatrix}$$ second order tensor

(c) $$a_{ij} \cdot a_{kj} = a_{i1}a_{k1} + a_{i2}a_{k2} + a_{i3}a_{k3} = A_{ik} = A_{ki} \quad (i, k = 1,2,3)$$

$$A_{11} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} = 1 + 1 + 1 = 3$$

$$A_{12} = a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = 0 + 4 + 2 = 6$$

$$A_{13} = a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} = 0 + 1 + 1 = 2$$

$$A_{21} = a_{21}a_{11} + a_{22}a_{12} + a_{23}a_{13} = A_{12} = 6$$

$$A_{22} = a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} = 0 + 16 + 4 = 20$$

$$A_{23} = a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0 + 4 + 2 = 6$$

$$A_{33} = a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33} = 0 + 1 + 1 = 2$$

$$A_{ik} = \begin{bmatrix} 3 & 6 & 2 \\ 6 & 20 & 6 \\ 2 & 6 & 2 \end{bmatrix}$$

second order tensor

(d) $a_{ij} b_j = a_{i1} b_1 + a_{i2} b_2 + a_{i3} b_3$

$$= \begin{Bmatrix} a_{11} b_1 + a_{12} b_2 + a_{13} b_3 \\ a_{21} b_1 + a_{22} b_2 + a_{23} b_3 \\ a_{31} b_1 + a_{32} b_2 + a_{33} b_3 \end{Bmatrix} = \begin{Bmatrix} 1+0+2 \\ 0+0+4 \\ 0+0+2 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 4 \\ 2 \end{Bmatrix}$$

(e) $c_{ij} \cdot d_{ij} = c_{i1} \cdot d_{i1} + c_{i2} d_{i2} + c_{i3} d_{i3} \quad , \quad i = 1, 2, 3$

$$= c_{11} d_{11} + c_{12} d_{12} + c_{13} d_{13}$$
$$+ c_{21} d_{21} + c_{22} d_{22} + c_{23} d_{23}$$
$$+ c_{31} d_{31} + c_{32} d_{32} + c_{33} d_{33}$$

$$= 0 + \frac{1}{4} + \frac{1}{4} - \frac{1}{4} + 0 + \frac{3}{4} - \frac{1}{4} - \frac{3}{4} + 0$$

$$= 0$$

2. (40 puan) Aşağıda (x_1, x_2, x_3) koordinatlarında verilen B tensörü için,

$$[B_{ij}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}$$

(a) Özdeğerleri bulunuz (10 puan)
(b) Özvektörleri bulunuz. (10 puan)
(c) Özvektörler birbirine diktir. Özvektörlerin oluşturduğu (x_1', x_2', x_3') sisteme gitmek için gerekli olan koordinat dönüşüm matrisini (a_{ij}) bulunuz. (10 puan)

$\{X'\} = [A]^T \{X\}$

	x_1	x_2	x_3
x_1'	a_{11}	a_{21}	a_{31}
x_2'	a_{12}	a_{22}	a_{32}
x_3'	a_{13}	a_{23}	a_{33}

$$a_{ij} = \cos(x_i, x_j') = \hat{e}_i \cdot \hat{e}_j'$$

(d) Bulduğunuz koordinat dönüşüm matrisini kullanarak $[B_{ij}]$ tensörünü (x_1', x_2', x_3') koordinat sisteminde yazınız. (10 puan)

(a) Invariants of B matrix

$\theta_1 = 2+3-3 = 2$

$$\theta_2 = \begin{vmatrix} 3 & 4 \\ 4 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = -25-6+6 = -25$$

$$\theta_3 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{vmatrix} = 2.(-25) = -50$$

$$-\lambda^3 + \theta_1 \lambda^2 - \theta_2 \lambda + \theta_3 = 0$$

$$-\lambda^3 + 2\lambda^2 + 25\lambda - 50 = 0$$

$\boxed{\lambda_1 = 5}$, $\boxed{\lambda_2 = 2}$, $\boxed{\lambda_3 = -5}$

$\boxed{\lambda_1 > \lambda_2 > \lambda_3}$

(b) for $\lambda_1 = 5$

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & 4 & -8 \end{bmatrix} \begin{Bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$-3 n_1^{(1)} = 0 \quad \Rightarrow \quad \boxed{n_1^{(1)} = 0}$$

$$-2 n_2^{(1)} + 4 n_3^{(1)} = 0 \quad \Rightarrow \quad n_2^{(1)} = 2 n_3^{(1)}$$

$$\left[\overset{0}{\cancel{n_1^{(1)}}}\right]^2 + \left[n_2^{(1)}\right]^2 + \left[n_3^{(1)}\right]^2 = 1$$

$$4\left[n_3^{(1)}\right]^2 + \left[n_3^{(1)}\right]^2 = 1 \quad \Rightarrow \quad \boxed{n_3^{(1)} = \mp \frac{1}{\sqrt{5}}} \quad \Rightarrow \quad \boxed{n_2^{(1)} = \mp \frac{2}{\sqrt{5}}}$$

$$\hat{n}_1 = 0\,\hat{e}_1 \mp \frac{2}{\sqrt{5}}\,\hat{e}_2 \mp \frac{1}{\sqrt{5}}\,\hat{e}_3$$

$$\boxed{\hat{n}_1 = \frac{1}{\sqrt{5}}\left(0\,\hat{e}_1 + 2\,\hat{e}_2 + 1\,\hat{e}_3\right)}$$

for $\lambda_1 = 2$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 4 & -5 \end{bmatrix} \begin{Bmatrix} n_1^{(2)} \\ n_2^{(2)} \\ n_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$n_2^{(2)} + 4 n_3^{(2)} = 0 \quad \Rightarrow \quad n_2^{(2)} = -4 n_3^{(2)}$$

$$4 n_2^{(2)} - 5 n_3^{(2)} = 0 \quad \Rightarrow \quad -16 n_3^{(2)} - 5 n_3^{(2)} = 0$$

$$-21 n_3^{(2)} = 0 \quad \Rightarrow \quad \boxed{n_3^{(2)} = 0}$$

$$\boxed{n_2^{(1)} = 0}$$

$$\left[n_1^{(2)}\right]^2 + \left[n_2^{(2)}\right]^2 + \left[n_3^{(2)}\right]^2 = 1 \quad \Rightarrow \quad \boxed{n_1^{(2)} = 1}$$

$$\boxed{\hat{n}_2 = 1\,\hat{e}_1 + 0\,\hat{e}_2 + 0\,\hat{e}_3}$$

for $\lambda_3 = -5$

$$\begin{bmatrix} 7 & 0 & 0 \\ 0 & 8 & 4 \\ 0 & 4 & 2 \end{bmatrix} \begin{Bmatrix} n_1^{(3)} \\ n_2^{(3)} \\ n_3^{(3)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$7 n_1^{(3)} = 0 \quad \Rightarrow \quad \boxed{n_1^{(3)} = 0}$$

$$8 n_2^{(3)} + 4 n_3^{(3)} = 0 \quad \Rightarrow \quad n_3^{(3)} = -2 n_2^{(3)}$$

$$[n_1^{(3)}]^2 + [n_2^{(3)}]^2 + [n_3^{(3)}]^2 = 1$$

$$[n_2^{(3)}]^2 + 4 [n_2^{(3)}]^2 = 1$$

$$\boxed{n_2^{(3)} = \mp \frac{1}{\sqrt{5}}} \quad \Rightarrow \quad \boxed{n_3^{(3)} = \pm \frac{2}{\sqrt{5}}}$$

$$\hat{n}_3 = 0\hat{e}_1 \mp \frac{1}{\sqrt{5}}\hat{e}_2 \pm \frac{2}{\sqrt{5}}\hat{e}_3$$

$$\boxed{\hat{n}_3 = \frac{1}{\sqrt{5}} \left(0\hat{e}_1 \mp 1\hat{e}_2 \pm 2\hat{e}_3 \right)}$$

$\therefore$

$$\hat{n}_1 = 0\,\hat{e}_1 + \frac{2}{\sqrt{5}}\hat{e}_2 + \frac{1}{\sqrt{5}}\hat{e}_3$$

$$\hat{n}_2 = -1\hat{e}_1 + 0\,\hat{e}_2 + 0\,\hat{e}_3$$

$$\hat{n}_3 = 0\,\hat{e}_1 + \frac{1}{\sqrt{5}}\hat{e}_2 - \frac{2}{\sqrt{5}}\hat{e}_3$$

(c)

	x_1	x_2	x_3
x_1'	a_{11}	a_{21}	a_{31}
x_2'	a_{12}	a_{22}	a_{32}
x_3'	a_{13}	a_{23}	a_{33}

$$a_{ij} = \cos(x_i, x_j') = \hat{e}_i \cdot \hat{e}_j'$$

	x_1	x_2	x_3
x_1'	0	$2/\sqrt{5}$	$1/\sqrt{5}$
x_2'	1	0	0
x_3'	0	$1/\sqrt{5}$	$-2/\sqrt{5}$

$$\therefore [A]^T = \begin{bmatrix} 0 & 2/\sqrt{5} & 1/\sqrt{5} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$$

$$[A] = \begin{bmatrix} 0 & 1 & 0 \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \end{bmatrix}$$

Transformation rule

(d)

$$[B_{ij}'] = [A]^T [B_{ij}] [A]$$

$$= \begin{bmatrix} 0 & 2/\sqrt{5} & 1/\sqrt{5} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 10/\sqrt{5} & 5/\sqrt{5} \\ 2 & 0 & 0 \\ 0 & -5/\sqrt{5} & 10/\sqrt{5} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \end{bmatrix}$$

$$[B_{ij}'] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -5 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

3. (35 puan) $Ox_1'x_2'x_3'$ ileand the $Ox_1x_2x_3$ kartezyan koordinat sistemleri arasındaki açılar aşağıdaki tabloda verilmiştir

	x_1	x_2	x_3
x_1'	60°	45°	120°
x_2'	120°	45°	60°
x_3'	45°	90°	45°

(a) İki eksen takımı arasındaki dönüşüm(transformasyon) matrisini bulunuz ve bunun uygun bir ortagonal bir dönüşüm olduğunu gösteriniz. (20 puan)

(b) $x_1 + x_2 + x_3 = 1/\sqrt{2}$ denkleminin $Ox_1'x_2'x_3'$ koordinatlarındaki biçimini ($b_1x_1' + b_2x_2' + b_3x_3' = b$) bulunuz (15 puan)

(a) $\{x'\} = [A]^T\{x\}$

$\{x\} = [A]\{x'\}$

$$[A]^T = \begin{bmatrix} 1/2 & \sqrt{2}/2 & -1/2 \\ -1/2 & \sqrt{2}/2 & 1/2 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix} ; \quad [A] = \begin{bmatrix} 1/2 & -1/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -1/2 & 1/2 & \sqrt{2}/2 \end{bmatrix}$$

$$[A]^T[A] \overset{?}{=} I$$

$$\begin{bmatrix} 1/2 & \sqrt{2}/2 & -1/2 \\ -1/2 & \sqrt{2}/2 & 1/2 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -1/2 & 1/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

it is an orthogonal transformation.

$$\{x\} = [A]\{x'\}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 1/2 & -1/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -1/2 & 1/2 & \sqrt{2}/2 \end{bmatrix} \begin{Bmatrix} x_1' \\ x_2' \\ x_3' \end{Bmatrix}$$

$$x_1 = \frac{1}{2}x_1' - \frac{1}{2}x_2' + \frac{\sqrt{2}}{2}x_3' \longrightarrow (1)$$

$$x_2 = \frac{\sqrt{2}}{2}x_1' + \frac{\sqrt{2}}{2}x_2' + 0\,x_3' \longrightarrow (2)$$

$$x_3 = -\frac{1}{2}x_1' + \frac{1}{2}x_2' + \frac{\sqrt{2}}{2}x_3' \longrightarrow (3)$$

(1), (2) ve (3) numaralı denklemleri eklersek

$$x_1 + x_2 + x_3 = \frac{\sqrt{2}}{2}x_1' + \frac{\sqrt{2}}{2}x_2' + \sqrt{2}\,x_3'$$

$$1/\sqrt{2} = \frac{\sqrt{2}}{2}x_1' + \frac{\sqrt{2}}{2}x_2' + \sqrt{2}\,x_3'$$

$$1 = x_1' + x_2' + 2x_3'$$

$\therefore$ $b_1 = 1$

$b_2 = 1$

$b_3 = 2$

$b = 1$