TOBB EKONOMİ VE TEKNOLOJİ ÜNİVERSİTESİ

MAK506 THEORY OF ELASTICITY
SPRING 2009
Due date: March 30, 2009
HOMEWORK 5

1. (Pr. 6.24, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase) For the case of plane stress, let the stress components be defined in terms of the function $\phi = \phi(x_1, x_2)$, known as the Airy stress function, by the relation ships,

$$\sigma_{11} = \phi_{,22}, \qquad \sigma_{22} = \phi_{,11}, \qquad \sigma_{12} = -\phi_{,12} \qquad \text{——— (1)}$$

Show that ϕ must satisfy the biharmonic equation $\nabla^4\phi = 0$ and that, in the absence of body forces, the equilibrium equations are satisfied identically by these stress components. If $\phi = Ax_1^3x_2^2 - Bx_1^5$ where A and B are constants, determine the relationship between A and B for this to be a valid stress function

Equilibrium equation in the absense of body forces

$$\sigma_{ij,j} = 0 \qquad i,j = 1,2 \qquad \text{——— (2)}$$

$i=1$

$$\sigma_{11,1} + \sigma_{12,2} = 0$$

$$\phi_{,221} + (-\phi_{,122}) = 0$$

$$0 = 0 \quad \text{identically satisfied}$$

$i=2$

$$\sigma_{12,1} + \sigma_{22,2} = 0$$

$$-\phi_{,121} + \phi_{,112} = 0$$

$$0 = 0 \quad \text{identically satisfied}$$

$$\nabla^4\phi = \frac{\partial^4\phi}{\partial x_1^4} + 2\frac{\partial^4\phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4\phi}{\partial x_2^4} = 0$$

$$-120Bx_1 + 2(12Ax_1) = 0$$

$$24A\cancel{x_1} = 120B\cancel{x_1}$$

$$A = 5B$$

2. (Pr. 6.26, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase) Show that $\phi = x_1^4 x_2 + 4x_1^2 x_2^3 - x_2^5$ is a valid Airy stress function, that is, that $\nabla^4 \phi = 0$, and compute the stress tensor for this case assuming a state of plane strain with $\nu = 0.25$

$$\sigma_{11} = \frac{\partial^2 \phi}{\partial x_2^2} = 24 x_1^2 x_2 - 20 x_2^3$$

$$\sigma_{12} = -\frac{\partial^2 \phi}{\partial x_1 \partial x_2} = -4x_1^3 - 24 x_1 x_2^2$$

$$\sigma_{22} = \frac{\partial^2 \phi}{\partial x_1^2} = 12 x_1^2 x_2 + 8 x_2^3$$

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22}) = 0.25\left(24x_1^2 x_2 - 20x_2^3 + 12x_1^2 x_2 + 8x_2^3\right)$$

$$\sigma_{33} = 9x_1^2 x_2 - 3x_2^3$$

$$\sigma_{31} = \sigma_{32} = \sigma_{13} = \sigma_{23} = 0$$

$$[\sigma_{ij}] = \begin{bmatrix} 24x_1^2 x_2 - 20x_2^3 & -4x_1^3 - 24x_1 x_2^2 & 0 \\ -4x_1^3 - 24x_1 x_2^2 & 12x_1^2 x_2 + 8x_2^3 & 0 \\ 0 & 0 & 9x_1^2 x_2 - 3x_2^3 \end{bmatrix}$$

3. (Pr. 6.29, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase) Consider the Airy stress function

$$\phi_5 = D_5 x_1^2 x_2^3 + F_5 x_2^5$$

(a) Show that for this to be valid stress function, $F = -D_5 / 5$

(b) Construct the composite stress function

$$\phi = \phi_5 + \phi_3 + \phi_2$$

where

$$\phi = D_5\left(x_1^2 x_2^3 - \frac{1}{5}x_2^5\right) + \frac{1}{2}B_3 x_1^2 x_2 + \frac{1}{2}A_2 x_1^2$$

For this stress function show that the stress components are

$$\sigma_{11} = D_5\left(6x_1^2 x_2 - 4x_2^3\right)$$

$$\sigma_{22} = 2D_5 x_2^3 + B_3 x_2 + A_2$$

$$\sigma_{12} = -6D_5 x_1 x_2^2 - B_3 x_1$$

(a) $$\nabla^4 \phi = \frac{\partial^4 \phi}{\partial x_1^4} + 2\frac{\partial^4 \phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \phi}{\partial x_2^4} = 0$$

$$120 F_5 x_2 + 2(12) D_5 x_2 = 0$$

$$\boxed{F_5 = -\frac{D_5}{5}}$$

(b) $$\phi = D_5\left(x_1^2 x_2^3 - \frac{1}{5}x_2^5\right) + \frac{1}{2}B_3 x_1^2 x_2 + \frac{1}{2}A_2 x_1^2$$

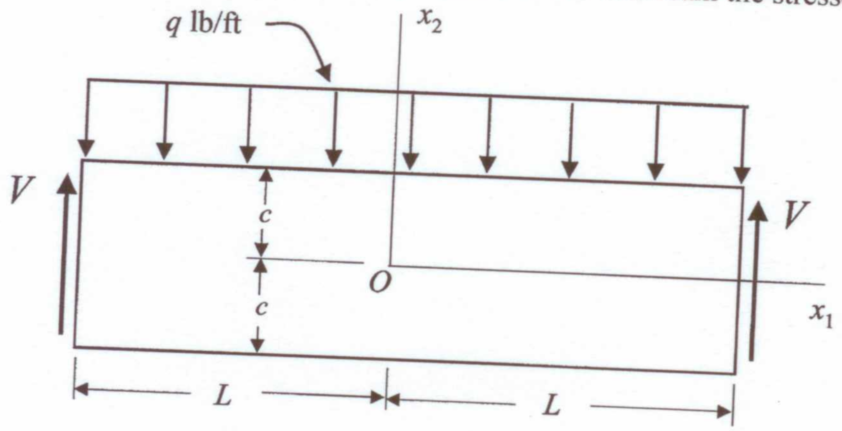
$$\sigma_{11} = \frac{\partial^2 \phi}{\partial x_2^2} \quad ; \quad \sigma_{22} = \frac{\partial^2 \phi}{\partial x_1^2} \quad ; \quad \sigma_{12} = -\frac{\partial^2 \phi}{\partial x_1 \partial x_2}$$

$$\sigma_{11} = D_5\left(6x_1^2 x_2 - 4x_2^3\right)$$

$$\sigma_{22} = 2D_5 x_2^3 + B_3 x_2 + A_2$$

$$\sigma_{12} = -6D_5 x_1 x_2^2 - B_3 x_1$$

4. (Pr. 6.30, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase) A rectangular beam of width unity and length $2L$ carries a uniformly distributed load of q lb/ft as shown below. Shear forces V support the beam at both ends. List the six boundary conditions for this beam the stresses must satisfy.



1.) $\sigma_{22} = -q$ @ $x_2 = +c$

2.) $\sigma_{22} = 0$ @ $x_2 = -c$

3) $\sigma_{12} = 0$ @ $x_2 = \pm c$

4.) $\int_{-c}^{+c} \sigma_{12} \, dx_2 = qL$ @ $x_1 = \mp L$

5.) $\int_{-c}^{+c} \sigma_{11} \, dx_2 = 0$

6.) $\int_{-c}^{+c} \sigma_{11} x_2 \, dx_2 = 0$ @ $x_1 = \mp L$

5. (Pr. 6.31, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase) Using boundary conditions 1, 2, and 3 listed in Problem 4 (Pr. 6.30, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase), show that the stresses in Problem 3 (Pr. 6.29, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase) require that

$$A_2 = -\frac{q}{2}; \quad B_3 = -\frac{3q}{4c}; \quad D_5 = \frac{q}{8c^3}$$

Thus, for the beam shown the stresses are

$$\sigma_{11} = \frac{q}{2I}\left(x_1^2 x_2 - \frac{2}{3}x_2^3\right)$$

$$\sigma_{22} = \frac{q}{2I}\left(\frac{1}{3}x_2^3 - c^2 x_2 - \frac{2}{3}c^3\right)$$

$$\sigma_{12} = -\frac{q}{2I}\left(x_1^2 x_2^2 + c^2 x_1\right)$$

where $I = \frac{2}{3}c^3$ is the plane moment of inertia of the beam cross section

$$\sigma_{22} = 2D_5 x_2^3 + B_3 x_2 + A_2 \qquad \text{——— (1)}$$

$$\sigma_{22}(x_1, c) = 2D_5 c^3 + B_3 c + A_2 = -q \qquad \text{——— (2)}$$

$$\sigma_{22}(x_1, -c) = -2D_5 c^3 - B_3 c + A_2 = 0 \qquad \text{——— (3)}$$

Adding (2) and (3)

$$2A_2 = -q \Rightarrow \boxed{A_2 = -\frac{q}{2}} \qquad \text{——— (4)}$$

$$\sigma_{12} = -6D_5 x_1 x_2^2 - B_3 x_1$$

$$\sigma_{12}(x_1, \pm c) = -6D_5 x_1 c^2 - B_3 x_1 = 0$$

$$-6D_5 c^2 - B_3 = 0 \Rightarrow \boxed{B_3 = -6D_5 c^2} \text{ — (5)}$$

$$\int_{-c}^{+c} \sigma_{12}\, dx_2 = \int_{-c}^{c}\left(-6D_5 x_1 x_2^2 - B_3 x_1\right) dx_2 = -6D_5 x_1 \frac{x_2^3}{3} - B_3 x_1 x_2 \Big|_{-c}^{c}$$

$$= x_1 \left[(-2D_5)(c^3 + c^3) - B_3(c + c)\right] = q \cdot L$$

$$= \cancel{L}\left[-4D_5 c^3 - 2c B_3\right] = q \cdot \cancel{L} \qquad \text{——— (6)}$$

Substitute (5) into (6)

$$-4D_5c^3 - 2c\cdot(-6D_5c^2) = q$$

$$8D_5c^3 = q \quad \Rightarrow \quad \boxed{D_5 = \frac{q}{8c^3}} \qquad \text{———} \quad (7)$$

Using (7) and (5)

$$B_3 = -6D_5c^2 = -6\cdot\frac{q}{8c^3}\cdot c^2 = -\frac{3q}{4c} \quad \Rightarrow \quad \boxed{B_3 = -\frac{3q}{4c}}$$

$$\sigma_{11} = D_5\left(6x_1^2x_2 - 4x_2^3\right)$$

$$= \frac{q}{2\cdot\frac{4c^3}{3}\cdot 3}\left(6x_1^2x_2 - 4x_2^3\right) = \frac{q}{12I}\left(6x_1^2x_2 - 4x_2^3\right)$$

$$\boxed{\sigma_{11} = \frac{q}{2I}\left(x_1^2x_2 - \frac{2}{3}x_2^3\right)}$$

$$\sigma_{22} = 2D_5x_2^3 + B_3x_2 + A_2$$

$$= 2\cdot\frac{q}{(4)\frac{2c^3}{3}\cdot 3}x_2^3 - \frac{3q}{4c}x_2 - \frac{q}{2}$$

$$\boxed{\sigma_{22} = \frac{q}{2I}\left(\frac{x_2^3}{3} - c^2x_2 - \frac{2c^3}{3}\right)}$$

$$\sigma_{12} = -6D_5x_1x_2^2 - B_3x_1$$

$$= -6\frac{q}{(4)\frac{2c^3}{3}\cdot 3}x_1x_2^2 - \frac{3q}{4c}x_1$$

$$\boxed{\sigma_{12} = -\frac{q}{2I}\left(x_1x_2^2 + c^2x_1\right)}$$

6. (Pr. 6.32, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase) Show that, using the stresses calculated in Problem 5 (Pr. 6.31, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase), the boundary conditions 4 and 5 are satisfied, but boundary condition 6 is not satisfied.

Boundary condition (4) has been already used in problem 5.

Boundary condition (5)

$$\int_{-c}^{c} \sigma_{11}\, dx_2 = \int_{-c}^{c} D_5 \left(6x_1^2 x_2 - 4x_2^3\right) dx_2 = D_5 \left(6x_1^2 \frac{x_2^2}{2} - 4\frac{x_2^4}{4}\right)\Bigg|_{-c}^{+c}$$

$$= 0 \qquad \text{automatically satisfied.}$$

Boundary condition (6)

$$\int_{-c}^{c} \sigma_{11}\, x_2\, dx_2 = \int_{-c}^{c} D_5 \left(6x_1^2 x_2^2 - 4x_2^4\right) dx_2 = D_5 \left(6x_1^2 \frac{x_2^3}{3} - 4\frac{x_2^5}{5}\right)\Bigg|_{-c}^{+c}$$

$$= D_5 \left(\frac{12}{3} x_1^2 c^3 - \frac{4}{5} \cdot 2c^5\right)$$

$$= D_5 \left(4x_1^2 c^3 - \frac{8}{5} c^5\right) \neq 0 \qquad \text{not satisfied}$$

7. (Pr. 6.33, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase) Continuing Problems 6.31 and 6.32, in order for boundary condition 6 to be satisfied and additional term is added to the stress function, namely

$$\phi_3 = D_3 x_2^3$$

Show that, from boundary condition 6,

$$D_3 = \frac{3q}{4c}\left(\frac{1}{15} - \frac{L^2}{6c^2}\right),$$

so that finally

$$\sigma_{11} = \frac{q}{2I}\left(x_1^2 - \frac{2}{3}x_2^2 + \frac{1}{15}c^2 - \frac{1}{6}L^3\right)x_2.$$

σ_{11} becomes

$$\sigma_{11} = D_5\left(6x_1^2 x_2 - 4x_2^3\right) + \frac{\partial^2 \phi_3}{\partial x_2^2}$$

$$\sigma_{11} = D_5\left(6x_1^2 x_2 - 4x_2^3\right) + 6 D_3 x_2$$

$$\int_{-c}^{c} \sigma_{11} x_2 \, dx_2 = D_5\left(4x_1^2 c^3 - \frac{8}{5}c^5\right) + 6D_3 \frac{x_2^3}{3}\Big|_{-c}^{c}$$

$$= D_5\left(4x_1^2 c^3 - \frac{8}{5}c^5\right) + 4D_3 c^3 = 0$$

$$D_3 = -\frac{D_5}{4c^3}\left(4x_1^2 c^3 - \frac{8}{5}c^5\right)$$

$$= -\frac{q}{(4)(8)c^3 \cdot c^3}\left(4x_1^2 c^3 - \frac{8}{5}c^5\right)$$

$$= -\frac{q}{8c}\left(\frac{L^2}{c^2} - \frac{2}{5}\right) = \frac{3q}{4c}\left(\frac{1}{15} - \frac{L^2}{6c^2}\right.$$

$$\sigma_{11} = \frac{q}{2I}\left[x_1^2 - \frac{2}{3}x_2^2 + \frac{1}{15}c^2 - \frac{1}{6}L^3\right]x_2$$