



# TOBB EKONOMİ VE TEKNOLOJİ ÜNİVERSİTESİ

## MAK506 THEORY OF ELASTICITY

SPRING 2009

Due date: March 12, 2009

### HOMEWORK 3

1. (Pr. 2.12, Advanced Strength and Applied Stress Analysis by Budynas) The state of stress at a point within a structure relative to an  $xyz$  coordinate system is given by,

$$[\sigma] = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 1 & 2 \\ -2 & 2 & -1 \end{bmatrix} \text{ kpsi}$$

Determine the normal and shear stresses and their directional cosines on a surface intersecting the point and parallel to the plane given by the equation

$$x + 2y - 2z = -4$$

An outward normal to the plane is given by

$$\vec{N} = \nabla(x + 2y - 2z + 4) = 1\hat{i} + 2\hat{j} - 2\hat{k}$$

$\therefore$  normal to the plane is found as

$$\vec{N}_i = 1\hat{i} + 2\hat{j} - 2\hat{k}$$

Unit normal vector is

$$\hat{n} = \frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k}$$

Traction vector on this plane is expressed as

$$T_i^n = \sigma_{ij} n_j \quad (\text{in kpsi})$$

$$T_1^n = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 = (3)\left(\frac{1}{3}\right) + (0)\left(\frac{2}{3}\right) + (-2)\left(-\frac{2}{3}\right) = \frac{7}{3}$$

$$T_2^n = \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3 = 0\left(\frac{1}{3}\right) + (1)\left(\frac{2}{3}\right) + (2)\left(-\frac{2}{3}\right) = -\frac{2}{3}$$

$$T_3^n = \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3 = (-2)\left(\frac{1}{3}\right) + (2)\left(\frac{2}{3}\right) + (-1)\left(-\frac{2}{3}\right) = \frac{4}{3}$$

$$|\vec{T}^n| = \sqrt{(T_1^n)^2 + (T_2^n)^2 + (T_3^n)^2} = \sqrt{\frac{49}{9} + \frac{4}{9} + \frac{16}{9}} = \frac{\sqrt{69}}{3} \text{ kpsi}$$

Normal stress

$$\begin{aligned}\sigma_{nn} &= \sigma_{ij} n_i n_j \\ &= \sigma_{11} n_1^2 + \sigma_{22} n_2^2 + \sigma_{33} n_3^2 \\ &\quad + 2(\sigma_{12} n_1 n_2 + \sigma_{13} n_1 n_3 + \sigma_{23} n_2 n_3) \\ &= -\frac{5}{9} \text{ kpsi}\end{aligned}$$

or

$$\begin{aligned}\sigma_{nn} &= \vec{T}^n \cdot \hat{n} = (T_1^n \hat{i} + T_2^n \hat{j} + T_3^n \hat{k}) \cdot \left(\frac{1}{3} \hat{i} + \frac{2}{3} \hat{j} - \frac{2}{3} \hat{k}\right) \\ &= \frac{7}{3} \cdot \frac{1}{3} + \left(-\frac{2}{3}\right) \left(\frac{2}{3}\right) + \left(\frac{4}{3}\right) \left(-\frac{2}{3}\right) = -\frac{5}{9}\end{aligned}$$

$$\sigma_{nt}^2 + \sigma_{nn}^2 = |\vec{T}^n|^2$$

$$\sigma_{nt}^2 + \frac{25}{81} = \frac{69}{9}$$

$$\sigma_{nt}^2 = \frac{596}{81} \Rightarrow \sigma_{nt} = 2.713 \text{ kpsi}$$

2. (Pr. 3.3, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase) The stress tensor at P relative to axes  $Px_1x_2x_3$  has components in MPa given by the matrix representation

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

where  $\sigma_{11}$  is unspecified. Determine a direction  $\hat{n}$  at P for which the plane perpendicular to  $\hat{n}$  will be stress-free, that is, for which  $t^{(\hat{n})} = 0$  on that plane. What is the required value of  $\sigma_{11}$  for this condition?

$$\sigma_{nn} = \vec{T}^n \cdot \hat{n} = 0$$

in order this to be true  $\vec{T}^n = t^{(\hat{n})} = 0$

$$\vec{T}^n = \sigma_{ij} n_j$$

$$T_1^n = \sigma_{11} n_1 + 2n_2 + 1n_3 = 0 \quad \text{————— (1)}$$

$$T_2^n = 2n_1 + 0n_2 + 2n_3 = 0 \quad \text{————— (2)}$$

$$T_3^n = 1n_1 + 2n_2 + 0n_3 = 0 \quad \Rightarrow \quad n_1 = -2n_2 \quad \text{————— (3)}$$

Substitute (3) into (2)

$$n_3 = 2n_2 \quad \text{————— (4)}$$

Substitute (3) and (4) into (1)

$$\sigma_{11}(-2n_2) + 2n_2 + 2n_2 = 0 \quad \Rightarrow \quad \sigma_{11} = 2 \text{ MPa}$$

Also

$$n_1^2 + n_2^2 + n_3^2 = 1 \quad \text{————— (5)}$$

Substitute (3) and (4) into (5)

$$4n_2^2 + n_2^2 + 4n_2^2 = 1 \quad \Rightarrow \quad n_2^2 = \frac{1}{9} \quad \Rightarrow \quad n_2 = \pm \frac{1}{3}$$

From (3)

$$n_1 = \mp \frac{2}{3}$$

From (4)

$$n_3 = \pm \frac{2}{3}$$

$$\hat{n} = \mp \frac{2}{3} \hat{e}_1 \pm \frac{1}{3} \hat{e}_2 \pm \frac{2}{3} \hat{e}_3$$

$$\begin{aligned} \hat{n}_1 &= -\frac{2}{3} \hat{e}_1 + \frac{1}{3} \hat{e}_2 + \frac{2}{3} \hat{e}_3 \\ \hat{n}_2 &= \frac{2}{3} \hat{e}_1 - \frac{1}{3} \hat{e}_2 - \frac{2}{3} \hat{e}_3 \end{aligned}$$

3. (Pr. 3.8, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase) Relative to cartesian axes  $Ox_1x_2x_3$  a stress field is given by the matrix

$$[\sigma_{ij}] = \begin{bmatrix} (1-x_1^2)x_2 + \frac{2}{3}x_2^3 & -(4-x_2^2)x_1 & 0 \\ -(4-x_2^2)x_1 & -\frac{1}{3}(x_2^3 - 12x_2) & 0 \\ 0 & 0 & (3-x_1^2)x_2 \end{bmatrix}$$

- (a) Show that the equilibrium equations are satisfied everywhere for zero body forces.  
 (b) Determine the stress vector at the point  $P(2, -1, 6)$  of the plane whose equation is  $3x_1 + 6x_2 + 2x_3 = 12$ .

(a)  $\sigma_{ij,j} + F_i = 0$

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = 0$$

$$-2x_1x_2 + 2x_1x_2 + 0 = 0 \quad \checkmark$$

$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} = 0$$

$$-(4-x_2^2) - x_2^2 + 4 + 0 = 0 \quad \checkmark$$

$$\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = 0$$

$$0 + 0 + 0 = 0 \quad \checkmark$$

(b) Outward normal to the plane

$$\vec{N} = \nabla(3x_1 + 6x_2 + 2x_3 - 12) = 3\hat{i} + 6\hat{j} + 2\hat{k}$$

unit normal vector is

$$\hat{n} = \frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} + \frac{2}{7}\hat{k}$$

at point  $P(2, -1, 6)$

$$[\sigma_{ij}] = \begin{bmatrix} 7/3 & -6 & 0 \\ -6 & -11/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_i^n = \sigma_{ij} \cdot n_j$$

$$T_1^n = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 = \frac{7}{3} \cdot \frac{3}{7} + (-6) \frac{6}{7} + 0 \left(\frac{2}{7}\right) = -\frac{29}{7}$$

$$T_2^n = \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3 = (-6) \frac{3}{7} + \left(-\frac{11}{3}\right) \left(\frac{6}{7}\right) + 0 \left(\frac{2}{7}\right) = -\frac{40}{7}$$

$$T_3^n = \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3 = 0 \left(\frac{3}{7}\right) + 0 \left(\frac{6}{7}\right) + (1) \frac{2}{7} = \frac{2}{7}$$

$$\vec{T}^n = -\frac{29}{7} \hat{i} - \frac{40}{7} \hat{j} + \frac{2}{7} \hat{k}$$

4. (Pr. 3.14, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase) When referred to principal axes at P, the stress matrix in ksi units is

$$[\sigma_{ij}^*] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

If the transformation matrix between the principal axes and axes  $Px_1x_2x_3$  is

$$[a_{ij}] = \frac{1}{\sqrt{2}} \begin{bmatrix} -\frac{3}{5} & 1 & -\frac{4}{5} \\ a_{21} & a_{22} & a_{23} \\ -\frac{3}{5} & -1 & -\frac{4}{5} \end{bmatrix}$$

Where  $a_{21}$ ,  $a_{22}$ , and  $a_{23}$  are to be determined, calculate  $[\sigma_{ij}]$ .

$$\hat{n}_1 = \frac{1}{\sqrt{2}} \left( -\frac{3}{5} \hat{e}_1 + \hat{e}_2 - \frac{4}{5} \hat{e}_3 \right)$$

$$\hat{n}_2 = \frac{1}{\sqrt{2}} \left( a_{21} \hat{e}_1 + a_{22} \hat{e}_2 + a_{23} \hat{e}_3 \right)$$

$$\hat{n}_3 = \frac{1}{\sqrt{2}} \left( -\frac{3}{5} \hat{e}_1 - \hat{e}_2 - \frac{4}{5} \hat{e}_3 \right)$$

$$\hat{n}_2 = \hat{n}_3 \times \hat{n}_1$$

$$= \frac{1}{\sqrt{2}} \left( -\frac{3}{5} \hat{e}_1 - \hat{e}_2 - \frac{4}{5} \hat{e}_3 \right) \times \frac{1}{\sqrt{2}} \left( -\frac{3}{5} \hat{e}_1 + \hat{e}_2 - \frac{4}{5} \hat{e}_3 \right)$$

$$= \frac{1}{2} \left( -\frac{3}{5} \hat{e}_3 - \frac{12}{25} \hat{e}_2 - \frac{3}{5} \hat{e}_3 + \frac{4}{5} \hat{e}_1 + \frac{12}{25} \hat{e}_2 + \frac{4}{5} \hat{e}_1 \right)$$

$$= \frac{1}{2} \left( \frac{8}{5} \hat{e}_1 + 0 \hat{e}_2 - \frac{6}{5} \hat{e}_3 \right)$$

$$\therefore \frac{a_{21}}{\sqrt{2}} = \frac{4}{5} \Rightarrow a_{21} = \frac{4\sqrt{2}}{5}$$

$$\frac{a_{22}}{\sqrt{2}} = 0 \Rightarrow a_{22} = 0$$

$$\frac{a_{23}}{\sqrt{2}} = -\frac{3}{5} \Rightarrow a_{23} = -\frac{3\sqrt{2}}{5}$$

$$[\sigma^*] = [A][\sigma][A^T] \quad (\text{According to Mase notation})$$

$$[A] = \frac{1}{\sqrt{2}} \begin{bmatrix} -\frac{3}{5} & 1 & -\frac{4}{5} \\ \frac{4\sqrt{2}}{5} & 0 & -\frac{3\sqrt{2}}{5} \\ -\frac{3}{5} & -1 & -\frac{4}{5} \end{bmatrix}$$

multiply both sides by  $[A]^T$

$$[A^T][\sigma^*] = \underbrace{[A^T][A]}_{[I]}[\sigma][A^T]$$

$$[A^T][\sigma^*] = [\sigma][A^T]$$

multiply by  $[A]$

$$[A^T][\sigma^*][A] = [\sigma] \underbrace{[A^T][A]}_{[I]}$$

$$[A^T][\sigma^*][A] = [\sigma]$$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} -\frac{3}{5} & \frac{4\sqrt{2}}{5} & -\frac{3}{5} \\ 1 & 0 & -1 \\ -\frac{4}{5} & -\frac{3\sqrt{2}}{5} & -\frac{4}{5} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 12 \end{bmatrix} \begin{bmatrix} -\frac{3}{5} & 1 & -\frac{4}{5} \\ \frac{4\sqrt{2}}{5} & 0 & -\frac{3\sqrt{2}}{5} \\ -\frac{3}{5} & -1 & -\frac{4}{5} \end{bmatrix} = [\sigma]$$

$$[\sigma] = \begin{bmatrix} 7 & 3 & 0 \\ 3 & 7 & 4 \\ 0 & 4 & 7 \end{bmatrix}$$