



MAK506 THEORY OF ELASTICITY
SPRING 2009
HOMEWORK 2

1. (Pr. 2.21, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase) For the matrix representation of tensor B shown below,

$$[B_{ij}] = \begin{bmatrix} 17 & 0 & 0 \\ 0 & -23 & 28 \\ 0 & 28 & 10 \end{bmatrix}$$

Determine the principal values (eigenvalues) and the principal directions (eigenvectors) of the tensor

The invariants of the matrix B

$$\theta_1 = 17 - 23 + 10 = 4$$

$$\theta_2 = \begin{vmatrix} -23 & 28 \\ 28 & 10 \end{vmatrix} + \begin{vmatrix} 17 & 0 \\ 0 & 10 \end{vmatrix} + \begin{vmatrix} 17 & 0 \\ 0 & -23 \end{vmatrix}$$

$$= -1014 + 170 - 391 = -1235$$

$$\theta_3 = \begin{vmatrix} 17 & 0 & 0 \\ 0 & -23 & 28 \\ 0 & 28 & 10 \end{vmatrix} = 17 \begin{vmatrix} -23 & 28 \\ 28 & 10 \end{vmatrix} = -17238$$

Eigenvalues can be found from the characteristic equation:

$$-\lambda^3 + \theta_1 \lambda^2 - \theta_2 \lambda + \theta_3 = 0$$

$$-\lambda^3 + 4\lambda^2 + 1235\lambda - 17238 = 0$$

$$\therefore \lambda_1 = 26, \lambda_2 = 17, \lambda_3 = -39$$

Consider the eigenvector $\vec{n}^{(1)}$ for $\lambda_1 = 26$

$$\vec{n}^{(1)} = n_1^{(1)} \hat{e}_1 + n_2^{(1)} \hat{e}_2 + n_3^{(1)} \hat{e}_3$$

$$\begin{bmatrix} 17-26 & 0 & 0 \\ 0 & -23-26 & 28 \\ 0 & 28 & 10-26 \end{bmatrix} \begin{Bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$



$$-9 n_1^{(1)} = 0 \Rightarrow n_1^{(1)} = 0$$

$$-49 n_2^{(1)} + 28 n_3^{(1)} = 0 \Rightarrow n_3^{(1)} = \frac{49}{28} n_2^{(1)} = \frac{7}{4} n_2^{(1)}$$

Also

$$[n_1^{(1)}]^2 + [n_2^{(1)}]^2 + [n_3^{(1)}]^2 = 1$$

$$[n_2^{(1)}]^2 + \frac{49}{16} [n_2^{(1)}]^2 = 1$$

$$n_2^{(1)} = \frac{4}{\sqrt{65}}$$

$$n_3^{(1)} = \frac{7}{\sqrt{65}}$$

$$\vec{n}^{(1)} = 0 \hat{e}_1 + \frac{4}{\sqrt{65}} \hat{e}_2 + \frac{7}{\sqrt{65}} \hat{e}_3 = \frac{4}{\sqrt{65}} \left(0 \hat{e}_1 + 1 \hat{e}_2 + \frac{7}{4} \hat{e}_3 \right)$$

Similarly,

$$\vec{n}^{(2)} = 1 \hat{e}_1 + 0 \hat{e}_2 + 0 \hat{e}_3 \quad \text{for } \lambda_2 = 17$$

$$\vec{n}^{(3)} = 0 \hat{e}_1 - \frac{7}{\sqrt{65}} \hat{e}_2 + \frac{4}{\sqrt{65}} \hat{e}_3 = \frac{4}{\sqrt{65}} \left(0 \hat{e}_1 - \frac{7}{4} \hat{e}_2 + 1 \hat{e}_3 \right)$$

→
for $\lambda_3 = -39$



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2. (Pr. 2.23, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase) Determine the principal values of the matrix

$$[K_{ij}] = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 11 & -\sqrt{3} \\ 0 & -\sqrt{3} & 9 \end{bmatrix}$$

and show that the principal axes $Ox_1^*x_2^*x_3^*$ are obtained from $Ox_1x_2x_3$ by a rotation of 60° about the x_1 axis

The invariants of the matrix K_{ij}

$$\theta_1 = 4 + 11 + 9 = 24$$

$$\theta_2 = \begin{vmatrix} 11 & -\sqrt{3} \\ -\sqrt{3} & 9 \end{vmatrix} + \begin{vmatrix} 4 & 0 \\ 0 & 9 \end{vmatrix} + \begin{vmatrix} 4 & 0 \\ 0 & 11 \end{vmatrix}$$

$$= 99 - 3 + 36 + 44 = 176$$

$$\theta_3 = \begin{vmatrix} 4 & 0 & 0 \\ 0 & 11 & -\sqrt{3} \\ 0 & -\sqrt{3} & 9 \end{vmatrix} = 4 \cdot (96) = 384$$

$$-\lambda^3 + \theta_1 \lambda^2 - \theta_2 \lambda + \theta_3 = 0$$

$$-\lambda^3 + 24\lambda^2 - 176\lambda + 384 = 0$$

$$\lambda_1 = 4, \quad \lambda_2 = 8, \quad \lambda_3 = 12$$

Consider the eigenvector $\vec{n}^{(2)}$ for $\lambda_2 = 8$

$$\begin{bmatrix} 4-8 & 0 & 0 \\ 0 & 11-8 & -\sqrt{3} \\ 0 & -\sqrt{3} & 9-8 \end{bmatrix} \begin{Bmatrix} n_1^{(2)} \\ n_2^{(2)} \\ n_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$



$$-4 n_1^{(2)} = 0 \Rightarrow n_1^{(2)} = 0$$

$$3 n_2^{(2)} - \sqrt{3} n_3^{(2)} = 0 \Rightarrow n_3^{(2)} = \frac{3}{\sqrt{3}} n_2^{(2)}$$

Also

$$[n_1^{(2)}]^2 + [n_2^{(2)}]^2 + [n_3^{(2)}]^2 = 1$$

$$[n_2^{(2)}]^2 + 3 [n_2^{(2)}]^2 = 1$$

$$n_2^{(2)} = \frac{1}{2} \Rightarrow n_3^{(2)} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

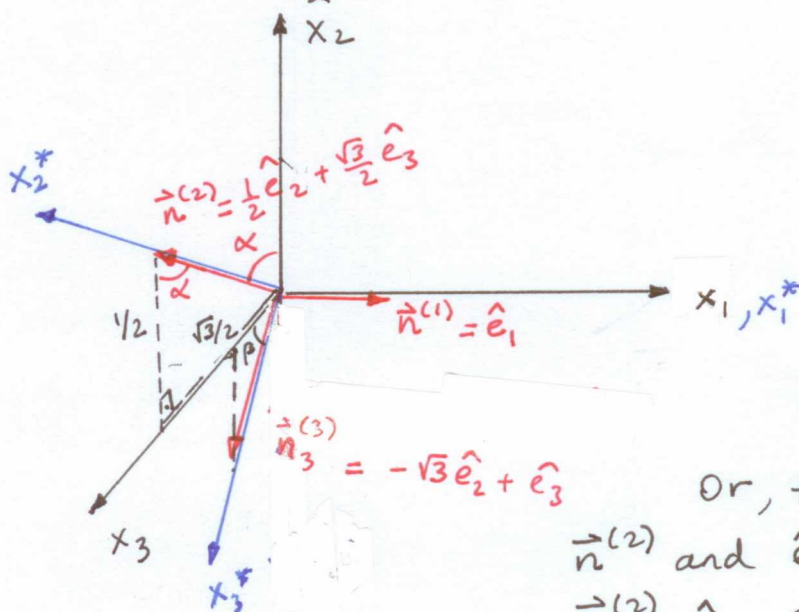
$$\vec{n}^{(2)} = 0 \hat{e}_1 + \frac{1}{2} \hat{e}_2 + \frac{\sqrt{3}}{2} \hat{e}_3 = \frac{\sqrt{3}}{2} \left(0 \hat{e}_1 + \frac{1}{\sqrt{3}} \hat{e}_2 + \hat{e}_3 \right)$$

similarly for $\lambda_1 = 4$

$$\vec{n}^{(1)} = 1 \hat{e}_1 + 0 \hat{e}_2 + 0 \hat{e}_3$$

and for $\lambda_3 = 12$

$$\vec{n}^{(3)} = 0 \hat{e}_1 - \frac{\sqrt{3}}{2} \hat{e}_2 + \frac{1}{2} \hat{e}_3 = \frac{1}{2} \left(0 \hat{e}_1 - \sqrt{3} \hat{e}_2 + \hat{e}_3 \right)$$



$$\tan \alpha = \frac{\sqrt{3}/2}{1/2}$$

$$\alpha = \tan^{-1}(\sqrt{3})$$

$$\alpha = 60^\circ$$

$$\tan \beta = \frac{\sqrt{3}}{1} \Rightarrow \beta = 60^\circ$$

Or, the angle between

$\vec{n}^{(2)}$ and \hat{e}_2 is

$$\vec{n}^{(2)} \cdot \hat{e}_2 = \cos \alpha = \frac{1}{2} \Rightarrow \alpha = 60^\circ$$

Angle between $\vec{n}^{(3)}$ and \hat{e}_3 is

$$\vec{n}^{(3)} \cdot \hat{e}_3 = \cos \alpha = \frac{1}{2} \Rightarrow \beta = 60^\circ$$



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3. (Pr. 2.24, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase) Determine the principal values $\lambda_{(q)}$ ($q=1,2,3$) and principal directions $\hat{n}^{(q)}$ ($q=1,2,3$) for the symmetric matrix

$$[T_{ij}] = \frac{1}{2} \begin{bmatrix} 3 & -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 9/2 & 3/2 \\ 1/\sqrt{2} & 3/2 & 9/2 \end{bmatrix}$$

Invariants of the matrix T_{ij}

$$\theta_1 = \frac{3}{2} + \frac{9}{4} + \frac{9}{4} = 6$$

$$\theta_2 = \begin{vmatrix} 9/4 & 3/4 \\ 3/4 & 9/4 \end{vmatrix} + \begin{vmatrix} 3/2 & 1/2\sqrt{2} \\ 1/2\sqrt{2} & 9/4 \end{vmatrix} + \begin{vmatrix} 3/2 & -1/2\sqrt{2} \\ -1/2\sqrt{2} & 9/4 \end{vmatrix}$$

$$\theta_2 = 11$$

$$\theta_3 = \begin{vmatrix} 3/2 & -1/2\sqrt{2} & 1/2\sqrt{2} \\ -1/2\sqrt{2} & 9/4 & 3/4 \\ 1/2\sqrt{2} & 3/4 & 9/4 \end{vmatrix} = 6$$

characteristic equation

$$-\lambda^3 + \theta_1 \lambda^2 - \theta_2 \lambda + \theta_3 = 0$$

$$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$$

$$\hat{n}^{(1)} = \frac{\sqrt{2}}{2} \hat{e}_1 - \frac{1}{2} \hat{e}_2 + \frac{1}{2} \hat{e}_3 = \frac{1}{2} (\sqrt{2} \hat{e}_1 - \hat{e}_2 + \hat{e}_3)$$

$$\hat{n}^{(2)} = 0 \hat{e}_1 + \frac{1}{\sqrt{2}} \hat{e}_2 + \frac{1}{\sqrt{2}} \hat{e}_3 = \frac{1}{\sqrt{2}} (0 \hat{e}_1 + \hat{e}_2 + \hat{e}_3)$$

$$\hat{n}^{(3)} = -\frac{\sqrt{2}}{2} \hat{e}_1 - \frac{1}{2} \hat{e}_2 + \frac{1}{2} \hat{e}_3 = \frac{1}{2} (-\sqrt{2} \hat{e}_1 - \hat{e}_2 + \hat{e}_3)$$