TOBB EKONOMİ VE TEKNOLOJİ ÜNİVERSİTESİ

MAK506 THEORY OF ELASTICITY
FALL 2010
Due date:11.10.2010
HOMEWORK 2

1. (Pr. 2.21, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase) For the matrix representation of tensor B shown below,

$$\left[B_{ij}\right]=\begin{bmatrix}17 & 0 & 0\\ 0 & -23 & 28\\ 0 & 28 & 10\end{bmatrix}$$

Determine the principal values (eigenvalues) and the principal directions (eigenvectors) of the tensor

$$\begin{bmatrix}17-\lambda & 0 & 0\\ 0 & -23-\lambda & 28\\ 0 & 28 & 10-\lambda\end{bmatrix}=0$$

$$\lambda_1=17$$
$$\lambda_2=26$$
$$\lambda_3=-39$$

$$\lambda_1=17\ durumu\,için,$$

$$\begin{bmatrix}17-17 & 0 & 0\\ 0 & -23-17 & 28\\ 0 & 28 & 10-17\end{bmatrix}\begin{pmatrix}n_x{}^{(1)}\\ n_y{}^{(1)}\\ n_z{}^{(1)}\end{pmatrix}=0$$

$$\left(n_x{}^{(1)}\right)^2+\left(n_y{}^{(1)}\right)^2+\left(n_z{}^{(1)}\right)^2=1$$

$$n_x{}^{(1)}=\pm 1$$

$$\overline{n}^{(1)}=\pm\hat{e}_1$$

$\lambda_1 = 26$ *durumu için,*

$$\begin{bmatrix} 17-26 & 0 & 0 \\ 0 & -23-26 & 28 \\ 0 & 28 & 10-26 \end{bmatrix} \begin{pmatrix} n_x^{(2)} \\ n_y^{(2)} \\ n_z^{(2)} \end{pmatrix} = 0$$

$$\left(n_x^{(2)}\right)^2 + \left(n_y^{(2)}\right)^2 + \left(n_z^{(2)}\right)^2 = 1$$

$$n_x^{(2)} = 0$$

$$n_z^{(2)} = \pm 0.868$$

$$n_y^{(2)} = \pm 0.496$$

$$\bar{n}^{(2)} = \pm 0.496\hat{e}_2 \pm 0.868\hat{e}_3$$

$\lambda_1 = -39$ *durumu için,*

$$\begin{bmatrix} 17+39 & 0 & 0 \\ 0 & -23+39 & 28 \\ 0 & 28 & 10+39 \end{bmatrix} \begin{pmatrix} n_x^{(2)} \\ n_y^{(2)} \\ n_z^{(2)} \end{pmatrix} = 0$$

$$\left(n_x^{(2)}\right)^2 + \left(n_y^{(2)}\right)^2 + \left(n_z^{(2)}\right)^2 = 1$$

$$n_x^{(3)} = 0$$

$$n_y^{(3)} = \pm 0.868$$

$$n_z^{(3)} = \pm 0.496$$

$$\bar{n}^{(3)} = \pm 0.868\hat{e}_2 \pm 0.496\hat{e}_3$$

2. (Pr. 2.23, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase) Determine the principal values of the matrix and show that the principal axes $Ox_1^* x_2^* x_3^*$ are obtained from $Ox_1 x_2 x_3$ by a rotation of 60° about the x_1 axis.

$$\left[K_{ij}\right] = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 11 & -\sqrt{3} \\ 0 & -\sqrt{3} & 9 \end{bmatrix}$$

$$\begin{bmatrix} 4-\lambda & 0 & 0 \\ 0 & 11-\lambda & -\sqrt{3} \\ 0 & -\sqrt{3} & 9-\lambda \end{bmatrix} = 0$$

$\lambda_1 = 4$

$\lambda_2 = 8$

$\lambda_3 = 12$

	x_1	x_2	x_3
$x_1^{'}$	Cos0	Cos90^{O}	Cos90^{O}
$x_2^{'}$	Cos90^{O}	Cos60^{O}	Cos30^{O}
$x_3^{'}$	Cos90^{O}	Cos150^{O}	Cos60^{O}

$$[A]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.866 \\ 0 & -0.866 & 0.5 \end{bmatrix}$$

$$[A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & -0.866 \\ 0 & 0.866 & 0.5 \end{bmatrix}$$

$$\left[K^{'}\right] = [A]^T [K][A]$$

$$\left[K^{'}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0,5 & 0,8660 \\ 0 & -0,8660 & 0,5 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 11 & -\sqrt{3} \\ 0 & -\sqrt{3} & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0,5 & -0,8660 \\ 0 & 0,8660 & 0,5 \end{bmatrix}$$

$$\left[K^{'}\right] = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

3. (Pr. 2.24, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase) Determine the principal values $\lambda_{(q)}$ $(q=1,2,3)$ and principal directions $\hat{\mathrm{n}}^{(q)}$ $(q=1,2,3)$ for the symmetric matrix

$$\left[T_{ij}\right]=\frac{1}{2}\begin{bmatrix} 3 & -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 9/2 & 3/2 \\ 1/\sqrt{2} & 3/2 & 9/2 \end{bmatrix}$$

$$Q_1=\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}=\frac{3}{2}+\frac{9}{4}+\frac{9}{4}=6$$

$$Q_2=\begin{vmatrix}\varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{yz} & \varepsilon_{zz}\end{vmatrix}+\begin{vmatrix}\varepsilon_{xx} & \varepsilon_{xz} \\ \varepsilon_{xz} & \varepsilon_{zz}\end{vmatrix}+\begin{vmatrix}\varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy}\end{vmatrix}=\left(\frac{9}{4}\right)^2-\left(\frac{3}{4}\right)^2+\frac{27}{8}-\frac{1}{8}+\frac{27}{8}-\frac{1}{8}=11$$

$$Q_3=\begin{vmatrix}\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz}\end{vmatrix}=\begin{vmatrix}\frac{3}{2} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{9}{4} & \frac{3}{4} \\ \frac{1}{2\sqrt{2}} & \frac{3}{4} & \frac{9}{4}\end{vmatrix}=6$$

$$-\lambda^3+\lambda^2 Q_1-\lambda Q_2+Q_3=0$$

$$-\lambda^3+6\lambda^2-11\lambda+6=0$$

$$\lambda_{(1)}=1$$

$$\lambda_{(2)}=2$$

$$\lambda_{(3)}=3$$

$$\lambda_{(1)}=1$$

$$\begin{bmatrix}\frac{3}{2}-1 & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{9}{4}-1 & \frac{3}{4} \\ \frac{1}{2\sqrt{2}} & \frac{3}{4} & \frac{9}{4}-1\end{bmatrix}\begin{pmatrix} n_x^{(1)} \\ n_y^{(1)} \\ n_z^{(1)}\end{pmatrix}=0$$

$$\left(n_x^{(1)}\right)^2+\left(n_y^{(1)}\right)^2+\left(n_z^{(1)}\right)^2=1$$

$$n_x^{(1)}=\pm 0.707$$

$$n_y^{(1)}=\pm 0.5$$

$$n_z^{(1)}=\pm 0.5$$

$$\overline{n}^{(1)}=\pm 0.707\hat{e}_1\pm 0.5\hat{e}_2\pm 0.5\hat{e}_3$$

$$\lambda_{(1)}=2$$

$$\begin{bmatrix} \frac{3}{2}-2 & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{9}{4}-2 & \frac{3}{4} \\ \frac{1}{2\sqrt{2}} & \frac{3}{4} & \frac{9}{4}-2 \end{bmatrix}\begin{pmatrix} n_x^{(2)} \\ n_y^{(2)} \\ n_z^{(2)} \end{pmatrix}=0$$

$$\left(n_x^{(2)}\right)^2+\left(n_y^{(2)}\right)^2+\left(n_z^{(2)}\right)^2=1$$

$$n_x^{(2)}=\pm 0.707$$

$$n_y^{(2)}=\pm 0.5$$

$$n_z^{(2)}=\pm 0.5$$

$$\overline{n}^{(2)}=\pm 0.707\hat{e}_1\pm 0.5\hat{e}_2\pm 0.5\hat{e}_3$$

$$\lambda_{(1)}=3$$

$$\begin{bmatrix} \frac{3}{2}-3 & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{9}{4}-3 & \frac{3}{4} \\ \frac{1}{2\sqrt{2}} & \frac{3}{4} & \frac{9}{4}-3 \end{bmatrix}\begin{pmatrix} n_x^{(3)} \\ n_y^{(3)} \\ n_z^{(3)} \end{pmatrix}=0$$

$$\left(n_x^{(3)}\right)^2+\left(n_y^{(3)}\right)^2+\left(n_z^{(3)}\right)^2=1$$

$$n_x^{(3)}=0$$

$$n_y^{(3)}=\pm 0.707$$

$$n_z^{(3)}=\pm 0.707$$

$$\bar{n}^{(3)}=\pm 0.707\hat{e}_2\pm 0.707\hat{e}_3$$

4. (Pr. 2-8, Elasticity-Tensor, Dyadic, and Engineering Approaches, P.C.Chou, N.J.Pagano)

   Given the following system of strains,

   $$\varepsilon_x=5+x^2+y^2+x^4+y^4$$

   $$\varepsilon_y=6+3x^2+3y^2+x^4+y^4$$

   $$\gamma_{xy}=10+4xy(x^2+y^2+2)$$

   $$\varepsilon_z=\gamma_{xz}=\gamma_{yz}=0$$

   Determine if the system of strains is possible. If this strain distribution is possible, find the displacement components in terms of x and y, assuming that the displacement and rotation at the origin are zero.

   First compatibility eq.

   $$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=2\frac{\partial^2\varepsilon_{xy}}{\partial x\partial y}$$

   $$\frac{\partial^2\varepsilon_x}{\partial y^2}=\frac{\partial^2\left(5+x^2+y^2+x^4+y^4\right)}{\partial y^2}=12y^2+2$$

   $$\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\left(6+3x^2+3y^2+x^4+y^4\right)}{\partial y^2}=12x^2+6$$

   $$\varepsilon_{xy}=\frac{\gamma_{xy}}{2}=\frac{10+4xy(x^2+y^2+2)}{2}=5+2xy\left(x^2+y^2+2\right)$$

   $$\frac{\partial^2\varepsilon_{xy}}{\partial x\partial y}=\frac{\partial^2\left(5+2xy\left(x^2+y^2+2\right)\right)}{\partial x\partial y}=6x^2+6y^2+4$$

   $$12y^2+12x^2+8=2\left(6x^2+6y^2+4\right)$$

Second compatibility eq.

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = 2\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} = \frac{\partial^2 \left(6 + 3x^2 + 3y^2 + x^4 + y^4\right)}{\partial z^2} = 0$$

$$\frac{\partial^2 \varepsilon_z}{\partial y^2} = 0$$

$$\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} = 0$$

Third compatibility eq.

$$\frac{\partial^2 \varepsilon_x}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xz}}{\partial z \partial x}$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} = 0$$

$$\frac{\partial^2 \varepsilon_x}{\partial z^2} = 0$$

$$\frac{\partial^2 \varepsilon_{xz}}{\partial z \partial x} = 0$$

Fourth compatibility eq.

$$\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = 0$$

$$\frac{\partial \varepsilon_{yz}}{\partial x} = 0$$

$$\frac{\partial \varepsilon_{zx}}{\partial y} = 0$$

$$\frac{\partial \varepsilon_{xy}}{\partial z} = 0$$

Fifth compatibility eq.

$$\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x}\right)$$

$$\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = 0$$

$$\frac{\partial \varepsilon_{zx}}{\partial y} = 0$$

$$\frac{\partial \varepsilon_{xy}}{\partial z} = 0$$

$$\frac{\partial \varepsilon_{yz}}{\partial x} = 0$$

Sixth compatibility eq.

$$\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y}\right)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = 0$$

$$\frac{\partial \varepsilon_{xy}}{\partial z} = 0$$

$$\frac{\partial \varepsilon_{yz}}{\partial x} = 0$$

$$\frac{\partial \varepsilon_{zx}}{\partial y} = 0$$

$$\varepsilon_x = \frac{\partial u_1}{\partial x} = 5 + x^2 + y^2 + x^4 + y^4$$

$$u_1(x, y, z) = 5x + \frac{x^3}{3} + xy^2 + \frac{x^5}{5} + xy^4 + c_1(y)$$

$$u_2(0,0,0) = 0,\ c_1(0) = 0$$

$$\varepsilon_y = \frac{\partial u_2}{\partial y} = 6 + 3x^2 + 3y^2 + x^4 + y^4$$

$$u_2\left(x, y, z\right) = 6y + 3x^2 y + y^3 + yx^4 + \frac{y^5}{5} + c_2\left(x\right)$$

$$u_2(0,0,0) = 0,\ \ c_2\left(0\right) = 0$$

$$\varepsilon_z = \frac{\partial u_3}{\partial z} = 0$$

$$u_2\left(x, y, z\right) = c_3(x, y)$$

$$u_2(0,0,0) = 0,\ \ c_3(0,0) = 0$$

$$u\left(x, y, z\right) = \left(5x + \frac{x^3}{3} + xy^2 + \frac{x^5}{5} + xy^4 + c_1\left(y\right)\right)\hat{e}_1 + \left(6y + 3x^2 y + y^3 + yx^4 + \frac{y^5}{5} + c_2\left(x\right)\right)\hat{e}_2$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{xy} = \frac{5x + \frac{x^3}{3} + xy^2 + \frac{x^5}{5} + xy^4 + c_1\left(y\right)}{\partial y} + \frac{6y + 3x^2 y + y^3 + yx^4 + \frac{y^5}{5} + c_2\left(x\right)}{\partial x}$$

$$\gamma_{xy} = 2xy + 4xy^3 + 6xy + 4x^3 y + c_1 + c_2 = 4xy\left(x^2 + y^2 + z^2\right) + c_1 + c_2$$

$$4xy\left(x^2 + y^2 + z^2\right) + c_1 + c_2 = 4xy\left(x^2 + y^2 + z^2\right) + 10 \ \textit{was given in problem}$$

$$c_1 + c_2 = 10$$