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**MAK506 THEORY OF ELASTICITY**
**SPRING 2009**
**HOMEWORK 1**

1. (Pr. 1.1, Elasticity, M. H. Sadd) For the given second- and first-order tensors

$$a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \qquad b_i = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

compute the following quantities: $a_{ii}$, $a_{ij}$, $a_{ij}a_{jk}$, $a_{ij}b_j$, $a_{ij}b_ib_j$, $b_ib_j$, $b_ib_i$. For each of the quantity, point out whether the result is a scalar, vector or a higher-order tensor.

(a) $\rightarrow$ $a_{ii} = a_{11} + a_{22} + a_{33} = 1 + 4 + 1 = 6 \rightarrow$ scalar
(zeroth order tensor)

(b) $\rightarrow$ $a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow$ second order tensor

(c) $\rightarrow$ $a_{ij}a_{jk} = a_{i1}a_{1k} + a_{i2}a_{2k} + a_{i3}a_{3k} = c_{ik}$

$i=1,\ k=1 \Rightarrow \quad c_{11} = a_{11}a_{11} + a_{12}a_{21} + a_{13}a_{31} = (1)(1) + (1)(0) + (1)(0) = 1$

$$c_{ik} = a_{ij} \cdot a_{jk} = \begin{bmatrix} 1 & 6 & 4 \\ 0 & 18 & 10 \\ 0 & 5 & 3 \end{bmatrix} \rightarrow \text{second order tensor}$$

(d) $\rightarrow$ $a_{ij} \cdot b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 = d_i$

$$d_i = a_{ij}b_j = \begin{Bmatrix} 3 \\ 4 \\ 2 \end{Bmatrix} \rightarrow \text{vector}$$
(first order tensor)

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(e) → $a_{ij} b_i b_j = a_{11} b_1 b_1 + a_{12} b_1 b_2 + a_{13} b_1 b_3$

$+ a_{21} b_2 b_1 + a_{22} b_2 b_2 + a_{23} b_2 b_3$

$+ a_{31} b_3 b_1 + a_{32} b_3 b_2 + a_{33} b_3 b_3 = 7$ → scalar

(f) → $b_i b_j = f_{ij} = \begin{bmatrix} b_1 b_1 & b_1 b_2 & b_1 b_3 \\ b_2 b_1 & b_2 b_2 & b_2 b_3 \\ b_3 b_1 & b_3 b_2 & b_3 b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}$

second order tensor

(g) → $b_i b_i = b_1 b_1 + b_2 b_2 + b_3 b_3$

$= b_1^2 + b_2^2 + b_3^2 = 1 + 0 + 4 = 5$ → scalar.

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2. (Pr. 2.5, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase) Expand the following expressions involving Kronecker deltas, and simplify where possible.
   (a) $\delta_{ij}\delta_{ij}$, (b) $\delta_{ij}\delta_{jk}\delta_{ik}$, (c) $\delta_{ij}\delta_{jk}$, (d) $\delta_{ij}A_{ik}$

(a) $\delta_{ij}\delta_{ij} = \delta_{i1}\delta_{i1} + \delta_{i2}\delta_{i2} + \delta_{i3}\delta_{i3}$ , $i=1,2,3$

$$= \delta_{11}\delta_{11} + \delta_{12}\delta_{12} + \delta_{13}\delta_{13}$$
$$+ \delta_{21}\delta_{21} + \delta_{22}\delta_{22} + \delta_{23}\delta_{23}$$
$$+ \delta_{31}\delta_{31} + \delta_{32}\delta_{32} + \delta_{33}\delta_{33} = 3$$

(b) $\delta_{ij}\delta_{jk}\delta_{ik}$

First find $\delta_{ij}\delta_{jk}$ which is part (c) of question 2.

$$\delta_{ij}\delta_{jk} = \delta_{i1}\delta_{1k} + \delta_{i2}\delta_{2k} + \delta_{i3}\delta_{3k} = [C_{ik}]_{3\times 3}$$

$i=1, k=1 \Rightarrow C_{11} = \delta_{11}\delta_{11} + \delta_{12}\delta_{21} + \delta_{13}\delta_{31} = 1$

$i=1, k=2 \Rightarrow C_{12} = \delta_{11}\delta_{12} + \delta_{12}\delta_{22} + \delta_{13}\delta_{32} = 0$

$i=1, k=3 \Rightarrow C_{13} = \delta_{11}\delta_{13} + \delta_{12}\delta_{23} + \delta_{13}\delta_{33} = 0$

$\therefore$

$$\delta_{ij}\delta_{jk} = [C_{ik}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \delta_{ik} \longrightarrow \text{part (c)}$$

$$\underbrace{\delta_{ij}\delta_{jk}}_{\delta_{ik}}\delta_{ik} = \delta_{ik}\delta_{ik} = \delta_{i1}\delta_{i1} + \delta_{i2}\delta_{i2} + \delta_{i3}\delta_{i3}$$
$$= \delta_{11}\delta_{11} + \cancel{\delta_{12}\delta_{12}} + \cancel{\delta_{13}\delta_{13}}$$
$$+ \cancel{\delta_{21}\delta_{21}} + \delta_{22}\delta_{22} + \cancel{\delta_{23}\delta_{23}}$$
$$+ \cancel{\delta_{31}\delta_{31}} + \cancel{\delta_{32}\delta_{32}} + \delta_{33}\delta_{33} = 3$$

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(c) $\Rightarrow$ $\delta_{ij}\delta_{jk} = \delta_{ik}$

(d) $\delta_{ij}A_{ik} = \delta_{1j}A_{1k} + \delta_{2j}A_{2k} + \delta_{3j}A_{3k} = [D_{jk}]_{3\times 3}$

$j=1,\ k=1 \Rightarrow D_{11} = \delta_{11}A_{11} + \cancel{\delta_{21}A_{21}} + \cancel{\delta_{31}A_{31}} = A_{11}$

$j=2,\ k=1 \Rightarrow D_{21} = \cancel{\delta_{12}A_{11}} + \delta_{22}A_{21} + \cancel{\delta_{32}A_{31}} = A_{21}$

$j=3,\ k=1 \Rightarrow D_{31} = \cancel{\delta_{13}A_{11}} + \cancel{\delta_{23}A_{21}} + \delta_{33}A_{31} = A_{31}$

$j=2,\ k=1 \Rightarrow D_{21} = \cancel{\delta_{12}A_{11}} + \delta_{22}A_{21} + \cancel{\delta_{32}A_{31}} = A_{21}$

$\therefore \quad D_{jk} = A_{jk}$

$$\boxed{\delta_{ij}A_{ik} = A_{jk}}$$

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3. (Pr. 2.19, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase) The angles between the respective axes of the $Ox_1'x_2'x_3'$ and the $Ox_1x_2x_3$ Cartesian systems are given by the table below

| | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| $x_1'$ | 45° | 90° | 45° |
| $x_2'$ | 60° | 45° | 120° |
| $x_3'$ | 120° | 45° | 60° |

Determine
(a) The transformation matrix between the two sets of axes, and show that it is a proper orthogonal transform.
(b) The equation of the plane $x_1 + x_2 + x_3 = 1/\sqrt{2}$ in its primed axes form, that is, in the form $b_1x_1' + b_2x_2' + b_3x_3' = b$

(a) $x_j' = a_{ij}x_i \Rightarrow \{x'\} = [A]^T\{x\}$

$$x_1' = a_{11}x_1 + a_{21}x_2 + a_{31}x_3$$
$$x_2' = a_{12}x_1 + a_{22}x_2 + a_{32}x_3$$
$$x_3' = a_{13}x_1 + a_{23}x_2 + a_{33}x_3$$

or

$x_i = a_{ij}x_j' \Rightarrow \{x\} = [A]\{x'\}$

$$x_1 = a_{11}x_1' + a_{12}x_2' + a_{13}x_3'$$
$$x_2 = a_{21}x_1' + a_{22}x_2' + a_{23}x_3'$$
$$x_3 = a_{31}x_1' + a_{32}x_2' + a_{33}x_3'$$

$$\therefore \quad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} \cos 45 & \cos 90 & \cos 45 \\ \cos 60 & \cos 45 & \cos 120 \\ \cos 120 & \cos 45 & \cos 60 \end{bmatrix}$$

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$$[A]^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 1/2 & \frac{\sqrt{2}}{2} & -1/2 \\ -1/2 & \frac{\sqrt{2}}{2} & 1/2 \end{bmatrix} \quad \therefore \quad \{x'\} = [A]^T\{x\}$$

$$[A] = \begin{bmatrix} \frac{\sqrt{2}}{2} & 1/2 & -1/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -1/2 & 1/2 \end{bmatrix} \quad \therefore \quad \{x\} = [A]\{x'\}$$

$$\therefore \quad x_1 = \frac{\sqrt{2}}{2}x_1' + \frac{1}{2}x_2' - \frac{1}{2}x_3'$$

$$x_2 = 0\,x_1' + \frac{\sqrt{2}}{2}x_2' + \frac{\sqrt{2}}{2}x_3'$$

$$+\quad x_3 = \frac{\sqrt{2}}{2}x_1' - \frac{1}{2}x_2' + \frac{1}{2}x_3'$$

$$\underbrace{x_1 + x_2 + x_3}_{\frac{1}{\sqrt{2}}} = \sqrt{2}\,x_1' + \frac{\sqrt{2}}{2}x_2' + \frac{\sqrt{2}}{2}x_3'$$

$$= \frac{\sqrt{2}}{2}\left(2x_1' + x_2' + x_3'\right)$$

$$\boxed{2x_1' + x_2' + x_3' = 1}$$

$$\therefore \quad b_1x_1' + b_2x_2' + b_3x_3' = b$$

$b_1 = 2$

$b_2 = 1$

$b_3 = 1$

$b = 1$