TOBB EKONOMİ VE TEKNOLOJİ ÜNİVERSİTESİ

MAK506 THEORY OF ELASTICITY
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HOMEWORK 1 Solution

1. (Pr. 1.1, Elasticity, M. H. Sadd) For the given second- and first-order tensors

$$a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \qquad b_i = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

compute the following quantities: a_{ii}, a_{ij}, $a_{ij}a_{jk}$, $a_{ij}b_j$, $a_{ij}b_ib_j$, b_ib_j, b_ib_i. For each of the quantity, point out whether the result is a scalar, vector or a higher-order tensor.

a)

$$a_{ii} = a_{11} + a_{22} + a_{33} = 1 + 4 + 1 = 6 \rightarrow scalar\ zeroth\ order\ tensor$$

b)

$$a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \text{sec}\,ond\ order\ tensor$$

c)

$$a_{ij}a_{jk} = a_{i1}a_{1k} + a_{i2}a_{2k} + a_{i3}a_{3k} = c_{ik}$$

$$i = 1, k = 1 \rightarrow \quad c_{11} = a_{11}a_{11} + a_{12}a_{21} + a_{13}a_{31} = (1)(1) + (1)(0) + (1)(0) = 1$$

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$$c_{ik} = a_{ij}a_{jk} = \begin{bmatrix} 1 & 6 & 4 \\ 0 & 18 & 10 \\ 0 & 5 & 3 \end{bmatrix} \rightarrow \text{sec}\,ond\ order\ tensor$$

d)

$$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 = d_i$$

$$d_i = a_{ij}b_j = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \rightarrow first\ order\ tensor(vector)$$

e)

$$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3$$
$$+ a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3$$
$$+ a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3 = 7 \rightarrow scalar$$

f)

$$b_ib_j = f_{ij} = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix} \rightarrow \text{sec}\,ond\ order\ tensor$$

g)

$$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3$$
$$= b_1^{\ 2} + b_2^{\ 2} + b_3^{\ 2} = 1 + 0 + 4 = 5 \rightarrow scalar$$

2. (Pr. 1.4, Elasticity, M. H. Sadd) Explicitly verify the following properties of the Kronecker delta:

$$\delta_{ij}a_j = a_i$$
$$\delta_{ij}a_{jk} = a_{ik}$$

$\delta_{ij}a_j = \delta_{i1}a_1 + \delta_{i2}a_2 + \delta_{i3}a_3 = d_i$

$i = 1,\ d_1 = a_1$

$i = 2,\ d_2 = a_2$

$i = 3,\ d_3 = a_3$

$\delta_{ij}a_j = d_i = a_i$

$\delta_{ij}a_{jk} = \delta_{i1}a_{1k} + \delta_{i2}a_{2k} + \delta_{i3}a_{3k} = c_{ik}$

$i = 1,\ \Rightarrow c_{1k} = a_{1k}$

$i = 2,\ \Rightarrow c_{2k} = a_{2k}$

$i = 3,\ \Rightarrow c_{3k} = a_{3k}$

$\delta_{ij}a_{jk} = c_{ik} = a_{ik}$

3. (Pr. 2.5, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase) Expand the following expressions involving Kronecker deltas, and simplify where possible.

(a) $\delta_{ij}\delta_{ij}$, (b) $\delta_{ij}\delta_{jk}\delta_{ik}$, (c) $\delta_{ij}\delta_{jk}$, (d) $\delta_{ij}A_{ik}$

a)

$$\delta_{ij}\delta_{ij} = \delta_{i1}\delta_{i1} + \delta_{i2}\delta_{i2} + \delta_{i3}\delta_{i3} \ , \ i = 1,2,3$$
$$= \delta_{11}\delta_{11} + \delta_{12}\delta_{12} + \delta_{13}\delta_{13}$$
$$+ \delta_{21}\delta_{21} + \delta_{22}\delta_{22} + \delta_{23}\delta_{23}$$
$$+ \delta_{31}\delta_{31} + \delta_{32}\delta_{32} + \delta_{33}\delta_{33} = 3$$

b)

$$\delta_{ij}\delta_{jk}\delta_{ik}$$

First find $\delta_{ij}\delta_{jk}$;

$$\delta_{ij}\delta_{jk} = \delta_{i1}\delta_{1k} + \delta_{i2}\delta_{2k} + \delta_{i3}\delta_{3k} = \left[c_{ik}\right]_{3x3}$$
$$i = 1,\ k = 1 \Rightarrow \quad c_{11} = \delta_{11}\delta_{11} + \delta_{12}\delta_{21} + \delta_{13}\delta_{31} = 1$$
$$i = 1,\ k = 2 \Rightarrow \quad c_{12} = \delta_{11}\delta_{12} + \delta_{12}\delta_{22} + \delta_{13}\delta_{32} = 0$$
$$i = 1,\ k = 3 \Rightarrow \quad c_{13} = \delta_{11}\delta_{13} + \delta_{12}\delta_{23} + \delta_{13}\delta_{33} = 0$$
$$\delta_{ij}\delta_{jk} = \left[c_{ik}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \delta_{ik}$$

$$\delta_{ij}\delta_{jk}\delta_{ik} = \delta_{ik}\delta_{ik} = \delta_{i1}\delta_{i1} + \delta_{i2}\delta_{i2} + \delta_{i3}\delta_{i3}$$
$$= \delta_{11}\delta_{11} + \delta_{12}\delta_{12} + \delta_{13}\delta_{13}$$
$$+ \delta_{21}\delta_{21} + \delta_{22}\delta_{22} + \delta_{23}\delta_{23}$$
$$+ \delta_{31}\delta_{31} + \delta_{32}\delta_{32} + \delta_{33}\delta_{33} = 3$$

c)

$$\delta_{ij}\delta_{jk} = \delta_{ik}$$

d)

$$\delta_{ij}A_{ik} = \delta_{1j}A_{1k} + \delta_{2j}A_{2k} + \delta_{3j}A_{3k} = \left[D_{jk}\right]_{3x3}$$
$$j = 1, k = 1 \Rightarrow D_{11} = \delta_{11}A_{11} + \delta_{21}A_{21} + \delta_{31}A_{31} = A_{11}$$
$$j = 2, k = 1 \Rightarrow D_{21} = \delta_{12}A_{11} + \delta_{22}A_{21} + \delta_{32}A_{31} = A_{21}$$
$$j = 3, k = 1 \Rightarrow D_{31} = \delta_{13}A_{11} + \delta_{23}A_{21} + \delta_{33}A_{31} = A_{31}$$
$$j = 2, k = 1 \Rightarrow D_{21} = \delta_{12}A_{11} + \delta_{22}A_{21} + \delta_{32}A_{31} = A_{21}$$
$$D_{jk} = A_{jk}$$
$$\delta_{ij}A_{ik} = A_{jk}$$

4. (Pr. 2.19, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase) The angles between the respective axes of the $Ox_1'x_2'x_3'$ and the $Ox_1x_2x_3$ Cartesian systems are given by the table below

	x_1	x_2	x_3
x_1'	45°	90°	45°
x_2'	60°	45°	120°
x_3'	120°	45°	60°

Determine

(a) The transformation matrix between the two sets of axes, and show that it is a proper orthogonal transform.

(b) The equation of the plane $x_1 + x_2 + x_3 = 1/\sqrt{2}$ in its primed axes form, that is, in the form $b_1x_1' + b_2x_2' + b_3x_3' = b$

a)

$$x_j' = a_{ij}x_i \Rightarrow \{x'\} = [A]^T\{x\}$$

$$x_1' = a_{11}x_1 + a_{21}x_2 + a_{31}x_3$$

$$x_2' = a_{12}x_1 + a_{22}x_2 + a_{32}x_3$$

$$x_3' = a_{13}x_1 + a_{23}x_2 + a_{33}x_3$$

or

$$x_i = a_{ij}x_j' \Rightarrow \{x\} = [A]\{x'\}$$

$$x_1 = a_{11}x_1' + a_{12}x_2' + a_{13}x_3'$$

$$x_2 = a_{21}x_1' + a_{22}x_2' + a_{23}x_3'$$

$$x_3 = a_{31}x_1' + a_{32}x_2' + a_{33}x_3'$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} \cos 45 & \cos 90 & \cos 45 \\ \cos 60 & \cos 45 & \cos 120 \\ \cos 120 & \cos 45 & \cos 60 \end{bmatrix}$$

$$[A]^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & \frac{\sqrt{2}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \end{bmatrix} \qquad \{x'\} = [A]^T \{x\}$$

$$[A] = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad \{x\} = [A]\{x'\} \qquad [A]^T[A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x_1 = \frac{\sqrt{2}}{2} x_1' + \frac{1}{2} x_2' - \frac{1}{2} x_3'$$

$$x_2 = (0) x_1' + \frac{\sqrt{2}}{2} x_2' + \frac{\sqrt{2}}{2} x_3'$$

$$x_3 = \frac{\sqrt{2}}{2} x_1' - \frac{1}{2} x_2' + \frac{1}{2} x_3'$$

$$x_1 + x_2 + x_3 = \frac{1}{\sqrt{2}} = \sqrt{2} x_1' + \frac{\sqrt{2}}{2} x_2' + \frac{\sqrt{2}}{2} x_3'$$

$$= \frac{\sqrt{2}}{2}\left(2x_1' + x_2' + x_3'\right)$$

$$2x_1' + x_2' + x_3' = 1$$

$$b_1 x_1' + b_2 x_2' + b_3 x_3' = b$$

$$b_1 = 2$$

$$b_2 = 1$$

$$b_3 = 1$$

$$b = 1$$