TOBB EKONOMİ VE TEKNOLOJİ ÜNİVERSİTESİ

Makina Mühendisliği Bölümü

MAK506 ELASTİSİTE TEORİSİ

Bahar Dönemi 2008-2009

Dönem Sonu (Final) sınavı

Dr. Mehmet Ali Güler

31 Mart 2009 Salı

Ad, Soyad ____________________

Öğrenci No ____________________

Verilen Zaman: **3 saat (12:30-15:30)**

Kitap ve Notlar Kapalı

- *Her soruyu dikkatle okuyun.*
- *Yaptığınız işlemleri gösterin.*
- *Sonuçları verilen kutular içine yazın.*
- *Temiz çalışın.*
- *Sınav salonunda cep telefonu kullanmak yasaktır.*

Soru No	Maksimum Puan	Puan
1	25	
2	25	
3	25	
4	25	
5	25	
Toplam	**125**	

Ön sayfa dahil, bu sınav kağıdında toplam **(6)** sayfa vardır.

1. **(25 points)** (Pr. 5.4.12, Elasticity in Engineering Mechanics by Boresi & Chong) The stress function for a cantilever beam loaded by a shear force P at the free end is

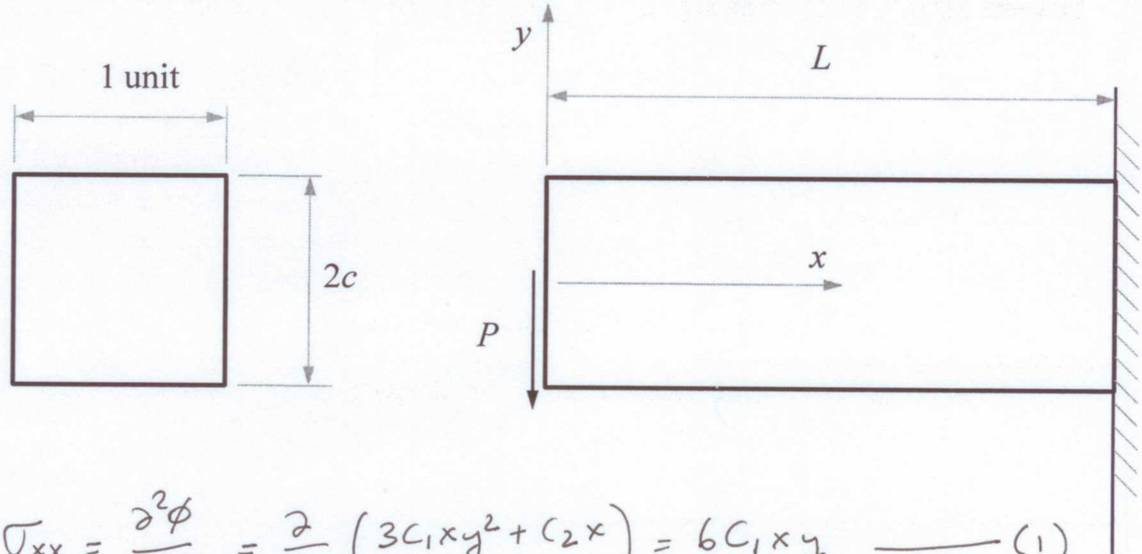
$$\phi = C_1 xy^3 + C_2 xy$$

(a) Evaluate the constants C_1 and C_2

(b) Derive the expressions for the displacement u and v

(c) Compare v with the expression derived for displacement y from the elementary beam theory, $EI\left(\frac{d^2y}{dx^2}\right) = M$.

Hint: write the boundary conditions at $x = 0$ and $x = L$ by considering force and moment equilibrium respectively.



$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial}{\partial y}\left(3C_1 xy^2 + C_2 x\right) = 6C_1 xy \quad \text{——— (1)}$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x}\left(C_1 y^3 + C_2 y\right) = 0 \quad \text{——— (2)}$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -\frac{\partial}{\partial y}\left(C_1 y^3 + C_2 y\right) = -3C_1 y^2 - C_2 \quad \text{——— (3)}$$

@ $x = 0$

$$P = \int_{-c}^{c} \sigma_{xy} \cdot (1)\, dy = \int_{-c}^{c} \left(-3C_1 y^2 - C_2\right) dy = -C_1 y^3 - C_2 y \Big|_{-c}^{c} \quad \text{——— (4)}$$

$$P = -2C_1 c^3 - 2C_2 c \quad \text{——— (5)}$$

@ $x = L$

$$\Sigma M\big|_{x=L} = P.L - \int_{-c}^{c} \sigma_{xx} \cdot y (1)\, dy = 0 \quad \text{——— (6)}$$

$$PL = \int_{-c}^{c} 6C_1 xy \cdot y\, dy \Big|_{x=L} = \int_{-c}^{c} 6C_1 xy^2 dy \Big|_{x=L} = 2C_1 L y^3 \Big|_{-c}^{c} = 4C_1 L c^3$$

$$P = 4C_1 c^3 \qquad (7)$$

$$\therefore \quad C_1 = \frac{P}{4c^3} \qquad (8)$$

using (8) and (5)

$$C_2 = \frac{3P}{4c} \qquad (9)$$

(6) $\quad \sigma_{xx} = 6C_1 xy \Rightarrow \boxed{\sigma_{xx} = \frac{3}{2}\frac{Pxy}{c^3}} \quad (10)$

$\sigma_{yy} = 0 \quad \boxed{\sigma_{yy} = 0} \quad (11)$

$\sigma_{xy} = -3C_1 y^2 - C_2 = -3\frac{P}{4c^3}y^2 - \frac{3P}{4c} \quad \boxed{\sigma_{xy} = -\frac{3P}{4c}\left(\frac{y^2}{c^2}+1\right)} \quad (12)$

Assume plane stress $\qquad (13)$

$\sigma_{zz} = 0$

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu \cancelto{0}{\sigma_{yy}} - \nu \cancelto{0}{\sigma_{zz}}\right) = \frac{\sigma_{xx}}{E} \qquad (14)$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{1}{E}\left(\frac{3}{2}\frac{Pxy}{c^3}\right) \Rightarrow u = \frac{3}{4}\frac{Px^2y}{Ec^3} + f(y) \qquad (15)$$

$$\varepsilon_{yy} = \frac{1}{E}\left(\cancelto{0}{\sigma_{yy}} - \nu\sigma_{xx} - \nu \cancelto{0}{\sigma_{zz}}\right) = -\frac{\nu}{E}\sigma_{xx}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = -\frac{\nu}{E}\,\frac{3}{2}\frac{Pxy}{c^3} \Rightarrow v = -\frac{3}{4}\frac{\nu P x y^2}{Ec^3} + g(x) \qquad (16)$$

$$\varepsilon_{xy} = \frac{1}{2\mu}\sigma_{xy} = \frac{1+\nu}{E}\left(-\frac{3P}{4c}\right)\left(\frac{y^2}{c^2}+1\right) = -\frac{1+\nu}{E}\frac{3P}{4c^3}\left(y^2+c^2\right) \qquad (17)$$

$$\mu = \frac{E}{2(1+\nu)}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left(\frac{3}{4}\frac{Px^2}{Ec^3} + f'(y) - \frac{3}{4}\frac{\nu P y^2}{Ec^3} + g'(x)\right) \quad \text{—}(18)$$

$$\varepsilon_{xy} = -\frac{1+\nu}{E}\frac{3P}{4c^3}\left(y^2+c^2\right) \quad \text{———}(19)$$

Equating (18) and (19)

$$\frac{3}{4}\frac{Px^2}{Ec^3} + f'(y) - \frac{3}{4}\frac{\nu P y^2}{Ec^3} + g'(x) = -\frac{1+\nu}{E}\frac{3P}{2c^3}\left(y^2+c^2\right)$$

$$g'(x) + \frac{3}{4}\frac{Px^2}{Ec^3} = -f'(y) + \frac{3}{4}\frac{\nu P y^2}{Ec^3} - \frac{1+\nu}{E}\frac{3P}{2c^3}\left(y^2+c^2\right) = K_1$$

$$\therefore \quad g'(x) + \frac{3}{4}\frac{Px^2}{Ec^3} = K_1 = \text{constant} \quad \text{———}(20)$$

Integrate in x

$$g(x) = -\frac{Px^3}{4Ec^3} + K_1 x + K_2 \quad \text{———}(21)$$

$$-f'(y) + \frac{3}{4}\frac{\nu P y^2}{Ec^3} - \frac{1+\nu}{E}\frac{3P}{2c^3}\left(y^2+c^2\right) = K_1$$

Integrate in y

$$f(y) = \frac{1}{4}\frac{\nu P y^3}{Ec^3} - \frac{1+\nu}{E}\frac{3P}{2c^3}\left(\frac{y^3}{3} + c^2 y\right) - K_1 y + K_3 \quad \text{—}(22)$$

From (15) & (16)

$$u = \frac{3}{4}\frac{Px^2 y}{Ec^3} + \frac{1}{4}\frac{\nu P y^3}{Ec^3} - \frac{1+\nu}{E}\frac{3P}{2c^3}\left(\frac{y^3}{3} + c^2 y\right) - K_1 y + K_3 \quad \text{———}(23)$$

$$v = -\frac{3}{4}\frac{\nu P x y^2}{Ec^3} - \frac{Px^3}{4Ec^3} + K_1 x + K_2 \quad \text{———}(24)$$

(c) Beam theory

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \quad \text{———} \quad (25)$$

$$I = \frac{1}{12}(1)(2c)^3 = \frac{2}{3}c^3 \quad \text{———} \quad (26)$$

$$M = -Px \quad \text{———} \quad (27)$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{-Px}{E \cdot \frac{2}{3}c^3} = -\frac{3}{2}\frac{Px}{Ec^3}$$

Integrate w.r.t x

$$\frac{dy}{dx} = -\frac{3}{4}\frac{Px^2}{Ec^3} + D_1 \quad \text{———} \quad (28)$$

Integrate

$$y = -\frac{Px^3}{4Ec^3} + D_1 x + D_2 \quad \text{———} \quad (29)$$

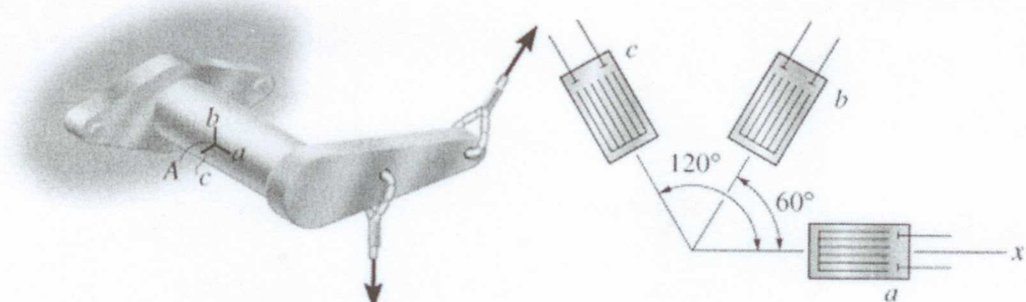
$\therefore$ with $D_1 = K_1$, $D_2 = K_2$ equation (29) & (24) are identical except for the ν term.

2. **(25 points)** (Example 10.8, Mechanics of Materials , R. C. Hibbeler) The state of strain at point A on the bracket in Fig. (a) is measured using the strain rosette shown in Fig. (b). Due to the loading, the readings from the gauges give displacements in an elastic material are given by $\varepsilon_a = 60(10^{-6})$, $\varepsilon_b = 135(10^{-6})$, and $\varepsilon_c = 264(10^{-6})$. Determine the in-plane principal strains at the point and the directions in which they act.

$$\varepsilon_{x'x'} = \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} + \frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\cos 2\theta + \left(\frac{\gamma_{xy}}{2}\right)\sin 2\theta$$

$$\varepsilon_{y'y'} = \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} - \frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\cos 2\theta - \left(\frac{\gamma_{xy}}{2}\right)\sin 2\theta$$

$$\frac{\gamma_{x'y'}}{2} = -\left(\frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\right)\sin 2\theta + \left(\frac{\gamma_{xy}}{2}\right)\cos 2\theta$$



(a) (b)

$$\varepsilon_a = \varepsilon_{xx} = 60(10^{-6}) \qquad \text{— (1)}$$

$$\varepsilon_b = \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} + \frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\overset{-1/2}{\cos 120} + \varepsilon_{xy}\overset{\sqrt{3}/2}{\sin 120} = 135(10^{-6}) \quad \text{— (2)}$$

$$\varepsilon_c = \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} + \frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\underbrace{\cos 240}_{-1/2} + \varepsilon_{xy}\underbrace{\sin 240}_{-\frac{\sqrt{3}}{2}} = 264(10^{-6} \quad \text{— (3)}$$

From (2) & (3)

$$\varepsilon_{xx}\left(\frac{1}{2}-\frac{1}{4}\right) + \varepsilon_{yy}\left(\frac{1}{2}+\frac{1}{4}\right) + \frac{\sqrt{3}}{2}\varepsilon_{xy} = 135(10^{-6}) \quad \text{— (4)}$$

$$\varepsilon_{xx}\left(\frac{1}{2}-\frac{1}{4}\right) + \varepsilon_{yy}\left(\frac{1}{2}+\frac{1}{4}\right) - \frac{\sqrt{3}}{2}\varepsilon_{xy} = 264(10^{-6}) \quad \text{— (5)}$$

From (4) & (5)

$$\varepsilon_{xx} + 3\varepsilon_{yy} + 2\sqrt{3}\,\varepsilon_{xy} = 540(10^{-6}) \qquad \text{— (6)}$$

$$\varepsilon_{xx} + 3\varepsilon_{yy} - 2\sqrt{3}\,\varepsilon_{xy} = 1056(10^{-6}) \qquad \text{— (7)}$$

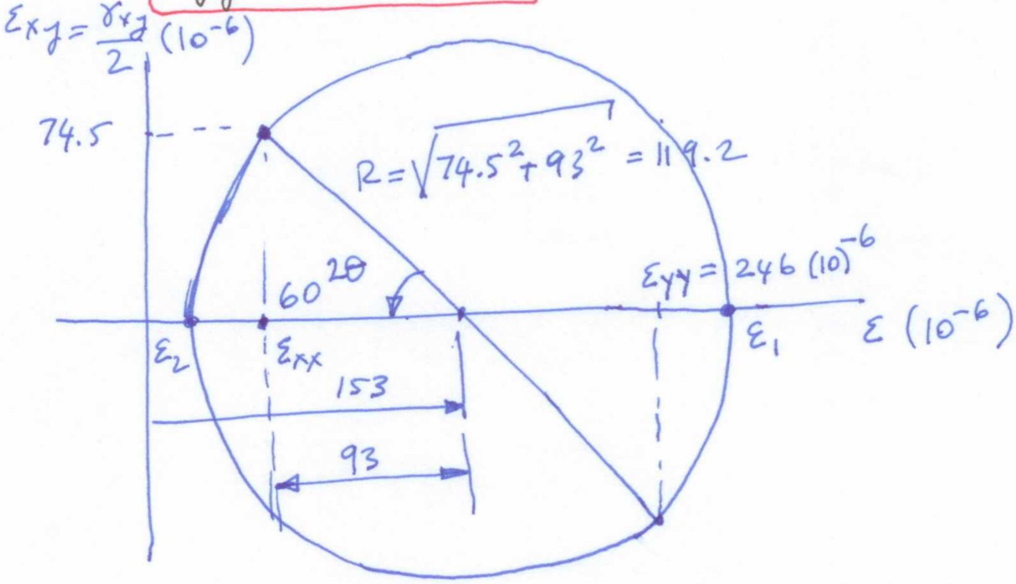
subtract (6) from (7)

$$-4\sqrt{3}\,\varepsilon_{xy} = 516\,(10^{-6}) \Rightarrow \boxed{-74.48\ 10^{-6} = \varepsilon_{xy}}$$

$$\boxed{\varepsilon_{xx} = 60\ 10^{-6}}$$

From (6)

$$\boxed{\varepsilon_{yy} = 246\ 10^{-6}}$$



$$R = \sqrt{(153-60)^2 + (74.5)^2}\ (10^{-6}) = 119.1\,(10^{-6})$$

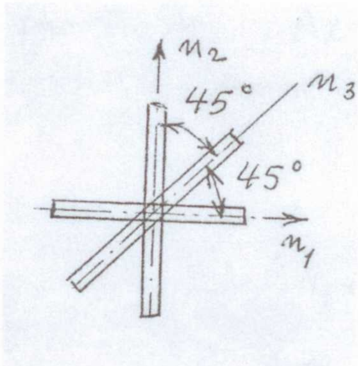
$$\varepsilon_1 = 153\,(10^{-6}) + 119.1\,(10^{-6}) = 272\,(10^{-6})$$

$$\varepsilon_2 = 153\,(10^{-6}) - 119.1\,(10^{-6}) = 33.9\,(10^{-6})$$

$$2\theta = \tan^{-1}\frac{74.5}{153-60} = 38.7$$

$$\theta = 19.3$$

3. **(25 points)** (Mech 408, Theory of Elasticity notes by Fazıl Erdoğan, Lehigh University) The normal strains in the directions n_1, n_2, n_3 are measured to be 2×10^{-3}, 4×10^{-3}, 1×10^{-3} respectively. Find the stress components referred to n_1, n_2 axes. Assume $E = 200$ GPa, $\nu = 0.3$



$$\varepsilon_{xy} = \frac{\gamma_{xy}}{2}$$

$$\varepsilon_{n1} = \varepsilon_{xx} = 2\times10^{-3} \quad \text{———— (1)}$$

$$\varepsilon_{n2} = \varepsilon_{yy} = 4\times10^{-3} \quad \text{———— (2)}$$

$$\varepsilon_{n3} = 1\times10^{-3} = \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} + \frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\cos 90 + \varepsilon_{xy}\sin 90$$

$$1\times10^{-3} = \frac{2+4}{2}\times10^{-3} + \varepsilon_{xy}$$

$$\varepsilon_{xy} = -2\times10^{-3} = \frac{\gamma_{xy}}{2} \Rightarrow \gamma_{xy} = -4\times10^{-3} \quad \text{— (3)}$$

Assume plane stress $\Rightarrow \sigma_{zz}=0$

$$\varepsilon_{xx} = 2\times10^{-3} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}) \quad \text{———— (4)}$$

$$\varepsilon_{yy} = 4\times10^{-3} = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx}) \Rightarrow \sigma_{yy} = E\varepsilon_{yy} + \nu\sigma_{xx} \quad \text{— (5)}$$

substitute (5) into (4)

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(E\varepsilon_{yy} + \nu\sigma_{xx})\right]$$

$$\varepsilon_{xx}(E) + \nu E\varepsilon_{yy} = \sigma_{xx}(1-\nu^2)$$

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left[\varepsilon_{xx} + \nu\varepsilon_{yy}\right] = 703.297 \text{ MPa}$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}\left[\varepsilon_{yy} + \nu\varepsilon_{xx}\right] = 1010.99 \text{ MPa}$$

$$\tau_{xy} = 2\mu\varepsilon_{xy} = 2\cdot\frac{E}{2(1+\nu)}\varepsilon_{xy} = -307.69 \text{ MPa}$$

4. **(25 points)** (Mech 408, Theory of Elasticity notes by Fazıl Erdoğan, Lehigh University) In an elastic solid the components of the displacement vector are given by

$$u_1 = K\left(1+2e^{x_1}+2x_2^2-3e^{-2x_3}\right)$$
$$u_2 = K\left(2+e^{2x_1}-4x_2^3-e^{-4x_3}\right)$$
$$u_3 = K\left(5+4e^{2x_1}+2e^{-3x_3}\right)$$

where K is a scaling constant.
(a) Find the strains
(b) At point ($x_1=1,\ x_2=2,\ x_3=0$) find the principal strains and principal directions.

(a) $\varepsilon_{11} = \dfrac{\partial u_1}{\partial x_1} = 2Ke^{x_1}$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) = \frac{K}{2}\left(4x_2+2e^{2x_1}\right) = K\left(2x_2+e^{2x_1}\right)$$

$$\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right) = \frac{K}{2}\left(6e^{-2x_3}+8e^{2x_1}\right) = K\left(3e^{-2x_3}+4e^{2x_1}\right)$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = K\left(-12x_2^2\right) = -12Kx_2^2$$

$$\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right) = \frac{K}{2}\left(4e^{-4x_3}\right) = 2Ke^{-4x_3}$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} = -6Ke^{-3x_3}$$

$$[\varepsilon_{ij}] = K\begin{bmatrix} (2e^{x_1}) & (2x_2+e^{2x_1}) & (3e^{-2x_3}+4e^{2x_1}) \\ (2x_2+e^{2x_1}) & (-12x_2^2) & (2e^{-4x_3}) \\ (3e^{-2x_3}+4e^{2x_1}) & (2e^{-4x_3}) & -6e^{-3x_3} \end{bmatrix}$$

(b) At point $x_1 = 1,\ x_2 = 2,\ x_3 = 0$

$$\varepsilon_{ij} = K \underbrace{\begin{bmatrix} 5.436 & 11.389 & 32.556 \\ 11.389 & -48 & 2 \\ 32.556 & 2 & -6 \end{bmatrix}}_{[\varepsilon'_{ij}]} = K\,\varepsilon'_{ij}$$

Principal strains

$$\left| K\varepsilon'_{ij} - \lambda I \right| = 0$$

or

$$\left| \varepsilon'_{ij} - \frac{\lambda}{K} I \right| = 0$$

let $\mu = \dfrac{\lambda}{K}$

$$\left| \varepsilon'_{ij} - \mu I \right| = 0$$

$$\begin{vmatrix} 5.436-\mu & 11.389 & 32.556 \\ 11.389 & -48-\mu & 2 \\ 32.556 & 2 & -6-\mu \end{vmatrix} = 0$$

Invariants

$$\theta_1 = 5.436 + (-48) + (-6) = -48.564$$

$$\theta_2 = \begin{vmatrix} 5.436 & 11.389 \\ 11.389 & -48 \end{vmatrix} + \begin{vmatrix} 5.436 & 32.556 \\ 32.556 & -6 \end{vmatrix} + \begin{vmatrix} -48 & 2 \\ 2 & -6 \end{vmatrix} = -1199.146$$

$$\theta_3 = \begin{vmatrix} 5.436 & 11.389 & 32.556 \\ 11.389 & -48 & 2 \\ 32.556 & 2 & -6 \end{vmatrix} = 54680.0716$$

$$-\mu^3 + \theta_1 \mu^2 - \theta_2 \mu + \theta_3 = 0$$

$$-\mu^3 - 48.564\mu^2 + 1199.146\mu + 54680.0716 = 0$$

$$\mu_1 = 34$$

$$\mu_2 = -31.481$$

$$\mu_3 = -51.084$$

For $\mu_1 = 34$

$$\hat{e}_1 = \begin{Bmatrix} 0.767 \\ 0.122 \\ 0.630 \end{Bmatrix}$$

For $\mu_2 = -31.481$

$$\hat{e}_2 = \begin{Bmatrix} -0.575 \\ -0.305 \\ 0.759 \end{Bmatrix}$$

For $\mu_3 = -51.084$

$$\hat{e}_3 = \begin{Bmatrix} 0.284 \\ -0.944 \\ -0.163 \end{Bmatrix}$$

5. **(25 points)** (Pr. 3.28, Continuum Mechanics for Engineers, G. Thomas Mase and George E. Mase) At point P in a continuum body, the stress tensor components are given in MPa with respect to axes $Px_1x_2x_3$ by the matrix

$$[\sigma_{ij}] = \begin{bmatrix} 1 & -3 & \sqrt{2} \\ -3 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 4 \end{bmatrix}$$

Determine

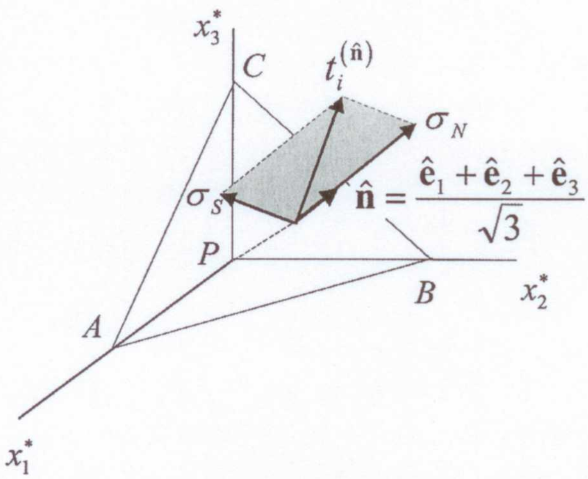
(a) The principal stress values σ_I, σ_{II}, and σ_{III}, together with the corresponding principal stress directions

(b) The stress invariants I_1, I_2, I_3

(c) The maximum shear stress value and the normal to the plane on which it acts

(d) The stress vector on the octahedral plane together with its normal and shear components

Note: Octohedral plane is shown in the Figure below.



(a), (b) $\quad I_1 = 1 + 1 + 4 = 6$

$$I_2 = \begin{vmatrix} 1 & -3 \\ -3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 4 \end{vmatrix} + \begin{vmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 4 \end{vmatrix}$$

$$= -8 + 2 + 2 = -4$$

$$I_3 = \begin{vmatrix} 1 & -3 & \sqrt{2} \\ -3 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 4 \end{vmatrix} = -24$$

$$-\sigma^3 + I_1\sigma^2 - I_2\sigma + I_3 = 0$$

$$-\sigma^3 + 6\sigma^2 + 4\sigma - 24 = 0 \quad \Rightarrow$$

$$\boxed{\begin{aligned} \sigma_I &= 6 \text{ MPa} \\ \sigma_{II} &= 2 \text{ MPa} \\ \sigma_{III} &= -2 \text{ MPa} \end{aligned}}$$

For $\sigma_I = 6$ MPa

$$\begin{bmatrix} 1-6 & -3 & \sqrt{2} \\ -3 & 1-6 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 4-6 \end{bmatrix} \begin{Bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$-5 n_1^{(1)} - 3 n_2^{(1)} + \sqrt{2}\, n_3^{(1)} = 0 \quad \text{——— (1)}$$

$$-3 n_1^{(1)} - 5 n_2^{(1)} - \sqrt{2}\, n_3^{(1)} = 0 \quad \text{——— (2)}$$

$$\sqrt{2}\, n_1^{(1)} - \sqrt{2}\, n_2^{(1)} - 2\, n_3^{(1)} = 0 \quad \text{——— (3)}$$

from (1)

$$n_3^{(1)} = \frac{5}{\sqrt{2}} n_1^{(1)} + \frac{3}{\sqrt{2}} n_2^{(1)} \quad \text{——— (4)}$$

Substitute (4) into (2)

$$-3n_1^{(1)} - 5n_2^{(1)} - 5\, n_1^{(1)} - 3n_2^{(1)} = 0$$

$$n_2^{(1)} = -n_1^{(1)} \quad \text{——— (5)}$$

$$\left[n_1^{(1)}\right]^2 + \left[n_2^{(1)}\right]^2 + \left[n_3^{(1)}\right]^2 = 1 \quad \text{——— (6)}$$

from (4) & (5)

$$n_3^{(1)} = \sqrt{2}\, n_1^{(1)} \quad \text{——— (7)}$$

Substitute (5) and (7) into (6)

$$\left[n_1^{(1)}\right]^2 + \left[n_1^{(1)}\right]^2 + 2\left[n_1^{(1)}\right]^2 = 1$$

$$n_1^{(1)} = \mp \frac{1}{2}$$

$$\boxed{n^{(1)} = \frac{1}{2}\left(\hat{e}_1 - \hat{e}_2 + \sqrt{2}\,\hat{e}_3\right)} \quad \text{——— (8)}$$

For $\sigma_{II} = 2$ MPa

$$-n_1^{(2)} - 3\,n_2^{(2)} + \sqrt{2}\,n_3^{(2)} = 0 \qquad (9)$$

$$-3n_1^{(2)} - n_2^{(2)} - \sqrt{2}\,n_3^{(2)} = 0 \qquad (10)$$

$$\sqrt{2}\,n_1^{(2)} - \sqrt{2}\,n_2^{(2)} + 2\,n_3^{(2)} = 0 \qquad (11)$$

from (9)

$$n_3^{(2)} = \frac{1}{\sqrt{2}}\,n_1^{(2)} + \frac{3}{\sqrt{2}}\,n_2^{(2)} \qquad (12)$$

Substitute (12) into (10)

$$-3\,n_1^{(2)} - n_2^{(2)} - n_1^{(2)} - 3\,n_2^{(2)} = 0$$

$$-4n_2^{(2)} = 4\,n_1^{(2)}$$

$$n_2^{(2)} = -\,n_1^{(2)} \qquad (13)$$

$$\left[n_1^{(2)}\right]^2 + \left[n_2^{(2)}\right]^2 + \left[n_3^{(2)}\right]^2 = 1 \qquad (14)$$

Substitute (13) into (12)

$$n_3^{(2)} = -\sqrt{2}\,n_1^{(2)} \qquad (15)$$

Substitute (15) and (13) into (14)

$$\left[n_1^{(2)}\right]^2 + \left[n_1^{(2)}\right]^2 + 2\left[n_1^{(2)}\right]^2 = 1$$

$$n_1^{(2)} = \mp\frac{1}{2}$$

$$\boxed{n^{(2)} = \frac{1}{2}\left(\hat{e}_1 - \hat{e}_2 - \sqrt{2}\,\hat{e}_3\right)} \qquad (16)$$

For $\sigma_{III} = -2\ \text{MPa}$

$$3n_1^{(3)} - 3n_2^{(3)} + \sqrt{2}\, n_3^{(3)} = 0 \tag{17}$$

$$-3n_1^{(3)} + 3n_2^{(3)} - \sqrt{2}\, n_3^{(3)} = 0 \tag{18}$$

$$\sqrt{2}\, n_1^{(3)} - \sqrt{2}\, n_2^{(3)} + 6\, n_3^{(3)} = 0 \tag{19}$$

from (17)

$$n_3^{(3)} = -\frac{3}{\sqrt{2}} n_1^{(3)} + \frac{3}{\sqrt{2}} n_2^{(3)} \tag{20}$$

Substitute (20) into (19)

$$\sqrt{2}\, n_1^{(3)} - \sqrt{2}\, n_2^{(3)} - \frac{18}{\sqrt{2}} n_1^{(3)} + \frac{18}{\sqrt{2}} n_2^{(3)} = 0$$

$$n_1^{(3)} = n_2^{(3)} \tag{21}$$

from (17)

$$n_3^{(3)} = 0 \tag{22}$$

$$\left[n_1^{(3)}\right]^2 + \left[n_2^{(3)}\right]^2 + \left[n_3^{(3)}\right]^2 = 1$$

$$\left[n_1^{(3)}\right]^2 + \left[n_1^{(3)}\right]^2 = 1$$

$$n_1^{(3)} = \frac{1}{\sqrt{2}}$$

$$\boxed{n^{(3)} = \frac{1}{\sqrt{2}}\left(\hat{e}_1 + \hat{e}_2\right)} \tag{23}$$

$$\{x'\} = [A]^T\{x\}$$

	x_1	x_2	x_3
x_1'	a_{11}	a_{21}	a_{31}
x_2'	a_{12}	a_{22}	a_{32}
x_3'	a_{13}	a_{23}	a_{33}

$$a_{ij} = \cos(x_i, x_j) = \hat{e}_i \cdot \hat{e}_j'$$

$$n^{(1)} = \frac{1}{2}\hat{e}_1 - \frac{1}{2}\hat{e}_2 + \frac{\sqrt{2}}{2}\hat{e}_3$$

$$n^{(2)} = \frac{1}{2}\hat{e}_1 - \frac{1}{2}\hat{e}_2 - \frac{\sqrt{2}}{2}\hat{e}_3$$

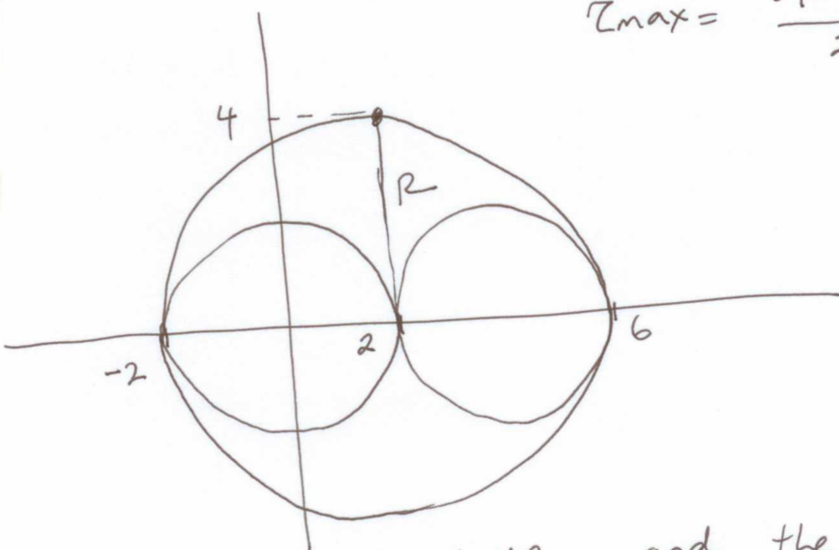
$$n^{(3)} = \frac{1}{\sqrt{2}}\hat{e}_1 + \frac{1}{\sqrt{2}}\hat{e}_2 + 0\,\hat{e}_3$$

	X_1	X_2	X_3
X_1'	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{2}}{2}$
X_2'	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$
X_3'	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0

$$\therefore \quad [A] = \begin{bmatrix} 1/2 & 1/2 & \frac{1}{\sqrt{2}} \\ -1/2 & -1/2 & 1/\sqrt{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

(c) $\sigma_I = 6\,MPa$, $\sigma_2 = 2\,MPa$, $\sigma_3 = -2\,MPa$

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{6-(-2)}{2} = 4\,MPa$$



Maximum shear stress is 4 MPa and the normal to the plane on which it acts can be found from the following analysis:

since $\sigma_1 > \sigma_2 > \sigma_3$ the unit normal from the principal axes is

$$\hat{n} = \frac{1}{\sqrt{2}}\hat{e}_1 + 0\,\hat{e}_2 + \frac{1}{\sqrt{2}}\hat{e}_3$$

$\therefore$

$$\hat{n}_{max} = [A]\cdot\{\hat{n}\} = \begin{bmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ -1/2 & -1/2 & 1/\sqrt{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{Bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{Bmatrix} = \begin{Bmatrix} \frac{1+\sqrt{2}}{2\sqrt{2}} \\ -\frac{1-\sqrt{2}}{2\sqrt{2}} \\ \frac{\sqrt{2}}{2\sqrt{2}} \end{Bmatrix}$$

$$\hat{n}_{max} = \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})\hat{e}_1 - (1-\sqrt{2})\hat{e}_2 + \sqrt{2}\,\hat{e}_3\right]$$

(d) The stress vector on the octohedral plane

$$\hat{n} = \frac{1}{\sqrt{3}}(\hat{e}_1 + \hat{e}_2 + \hat{e}_3)$$

$$\vec{T}^n = \sigma \cdot \hat{n} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{Bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{Bmatrix}$$

$$\vec{T}^n = \frac{\sigma_1}{\sqrt{3}}\hat{e}_1 + \frac{\sigma_2}{\sqrt{3}}\hat{e}_2 + \frac{\sigma_3}{\sqrt{3}}\hat{e}_3$$

$$= \frac{1}{\sqrt{3}}(6\hat{e}_1 + 2\hat{e}_2 - 2\hat{e}_3)$$

$$\sigma_N = \vec{T}^n \cdot \hat{n} = \left(\frac{6}{\sqrt{3}}\hat{e}_1 + \frac{2}{\sqrt{3}}\hat{e}_2 - \frac{2}{\sqrt{3}}\hat{e}_3\right) \cdot \left(\frac{1}{\sqrt{3}}\hat{e}_1 + \frac{1}{\sqrt{3}}\hat{e}_2 + \frac{1}{\sqrt{3}}\hat{e}_3\right)$$

$$= \frac{6}{3} + \frac{2}{3} - \frac{2}{3} \Rightarrow \quad \sigma_N = 2\text{ MPa}$$

Octohedral shear stress

$$\sigma_s^2 + \sigma_N^2 = |\vec{T}^n|^2$$

$$\sigma_s^2 + (2)^2 = \frac{36}{3} + \frac{4}{3} + \frac{4}{3}$$

$$\sigma_s^2 = \frac{44}{3} - 4 = \frac{32}{3}$$

$$\sigma_s = \sqrt{\frac{32}{3}}$$