TOBB EKONOMİ VE TEKNOLOJİ ÜNİVERSİTESİ

Makina Mühendisliği Bölümü

MAK501 MÜHENDİSLİK MATEMATİĞİ

Güz Dönemi 2014

Ara Sınav

Dr. Mehmet Ali Güler

20 Ekim 2014 Pazartesi

Ad, Soyad ______________________

Öğrenci No ______________________

Verilen Zaman: **2 saat (18:30-20:30)**

Soru No	Maksimum Puan	Puan
1	25	
2	25	
3	25	
4	25	
5	25	
Toplam	**125**	

ÖNEMLİ UYARI !!!

Yükseköğretim Kurumları Öğrenci Disiplin Yönetmeliği Madde 9-m'ye göre "sınavlarda kopya yapmak veya yaptırmak veya bunlara teşebbüs etmek" fiilinin suçu YÜKSEKÖĞRETİM KURUMUNDAN BİR VEYA İKİ YARIYIL İÇİN UZAKLAŞTIRMA cezasıdır.

Özel Sınav Kuralları:

Sınav süresince cep telefonları kapalı konumda olmak suretiyle sıra üzerine konulmalıdır. Kullanılacak formül kağıdı arkalı önlü bir A4 kağıdı olmalıdır.

UYARI VE KURALLARI OKUDUM.

Öğrencinin İmzası:

Ön sayfa dahil, bu sınav kağıdında toplam **(11)** sayfa vardır.

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Soru 1 (Fourier Series Expansion)

Aşağıdaki fonksiyon'un Fourier Seri açılımınını kullanarak, fonksiyonun N=1, 3 ve 5 için $x=2$ değerindeki sayısal değerlerini hesaplayınız. $N=1$ ve 3 için fonksiyonun Fourier serisinin grafiğini yaklaşık olarak çiziniz

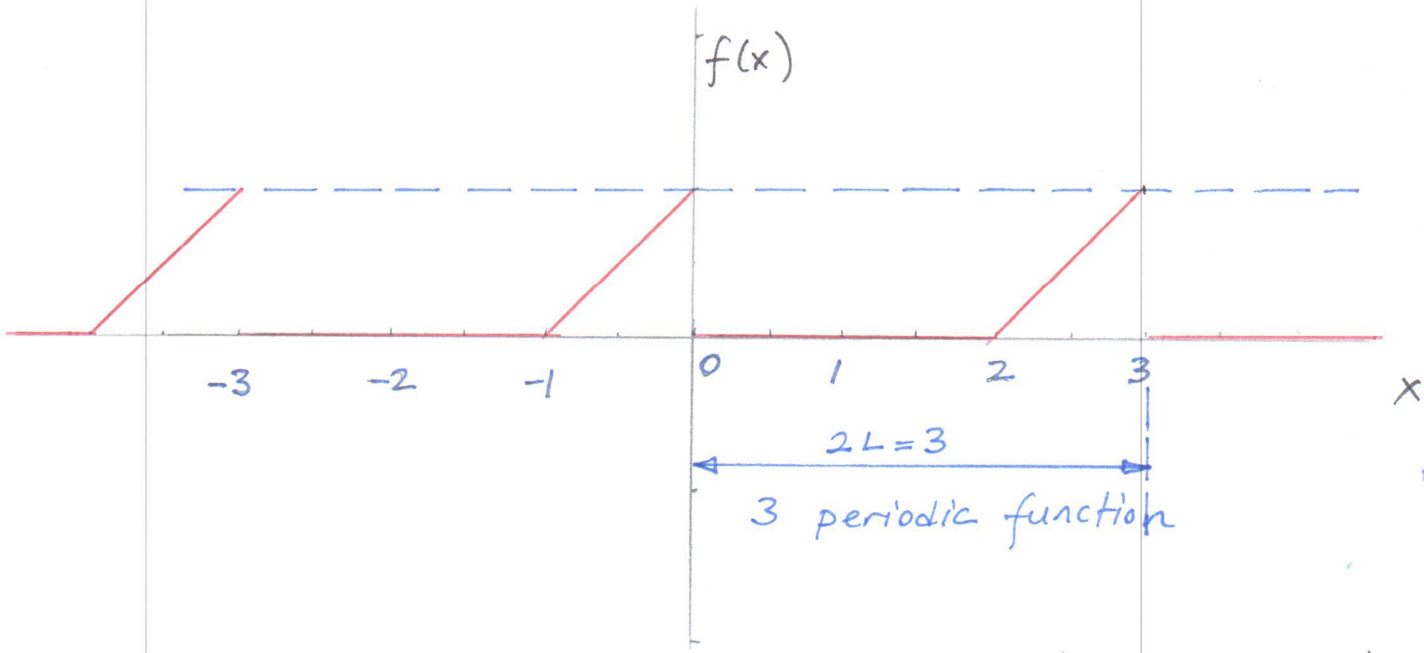
$$f(x)=\begin{cases} 0 & if\, 0\le x<2 \\ x-2 & if\, 2\le x<3 \end{cases}$$

Using the Fourier Series Expansion of the given function $f(x)$, find the value of $f(x)$ for N=1,3 & 5 at the value of $x=2$. Also sketch the graph of $f(x)$ for $N=1$ and 3 approximately

$$f(x)=\begin{cases} 0 & if\, 0\le x<2 \\ x-2 & if\, 2\le x<3 \end{cases}$$

Cevap: Function is $2L=3$ periodic : $(L=3/2)$

$$a_0=\frac{1}{2L}\int_{-L}^{L} f(x)\,dx=\frac{1}{2L}\int_{0}^{2L} f(x)\,dx=\frac{1}{3}\int_{0}^{3} f(x)\,dx=\frac{1}{3}\cdot\int_{2}^{3}(x-2)\,dx=\frac{1}{6}$$



Note that, One can take the stating and ending points of the period either $(0,3)$ or $(-3/2, 3/2)$ if you use $(-3/2, 3/2)$ $f(x)$ should be defined as

$$f(x)=\begin{cases} 0 & if\ -3/2\le x<-1 \\ x+1 & if\ -1\le x<0 \\ 0 & if\ 0\le x<3/2 \end{cases}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{0}^{2L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{3/2}\int_{0}^{3} f(x)\cos\frac{n\pi x}{3/2}\,dx = \frac{2}{3}\int_{2}^{3} \underbrace{(x-2)}_{u}\,\underbrace{\cos\frac{2n\pi x}{3}\,dx}_{dv}$$

$$= \frac{2}{3}\left[(x-2)\frac{3}{2n\pi}\sin\frac{2n\pi x}{3}\Big|_2^3 - \frac{3}{2n\pi}\int_2^3 \sin\frac{2n\pi x}{3}\,dx\right]$$

$$= \frac{2}{3}\left[\left(\frac{3}{2n\pi}\right)^2\cos\frac{2n\pi x}{3}\Big|_2^3\right] = \frac{3}{2n^2\pi^2}\left(\cos 2n\pi - \cos\frac{4n\pi}{3}\right)$$

$$a_n = \frac{3}{2n^2\pi^2}\left(1 - \cos\frac{4n\pi}{3}\right)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{0}^{2L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{3/2}\int_{0}^{3} f(x).\sin\frac{n\pi x}{3/2}\,dx = \frac{2}{3}\int_{2}^{3} \underbrace{(x-2)}_{u}\,\underbrace{\sin\frac{2n\pi x}{3}\,dx}_{dv}$$

$$= \frac{2}{3}\left[(x-2)\left(-\frac{3}{2n\pi}\right)\cos\frac{2n\pi x}{3}\Big|_2^3 - \left(-\frac{3}{2n\pi}\right)\int_2^3 \cos\frac{2n\pi x}{3}\,dx\right]$$

$$= \frac{2}{3}\left[\left(-\frac{3}{2n\pi}\right)\underbrace{\cos 2n\pi}_{1} + \left(\frac{3}{2n\pi}\right)^2 \sin\frac{2n\pi x}{3}\Big|_2^3\right]$$

$$= -\frac{1}{n\pi} - \frac{3}{2n^2\pi^2}\sin\frac{4n\pi}{3}$$

MIDTERM SOLUTION

$$f(x) = \frac{1}{6} + \sum_{n=1}^{\infty} \frac{3}{2n^2\pi^2}\left(1 - \cos\frac{4n\pi}{3}\right)\cos\frac{2n\pi x}{3}$$

$$+ \sum_{n=1}^{\infty}\left(-\frac{1}{n\pi} - \frac{3}{2n^2\pi^2}\sin\frac{4n\pi}{3}\right)\sin\frac{2n\pi x}{3}$$

$N=1 \Rightarrow f(2) = 0.21436$

$N=3 \Rightarrow f(2) = 0.01953$

$N=5 \Rightarrow f(2) = 0.00995$

Soru 2 (Quarter Range Sine (QRS) and Quarter Range Cosine (QRC) Expansion)

Aşağıdaki fonksiyon'un çeyrek sinüs ve çeyrek kosinüs açınımlarını türeterek çiziniz. Çiziminiz $-18 < x < 18$ aralığını kapsamalıdır.

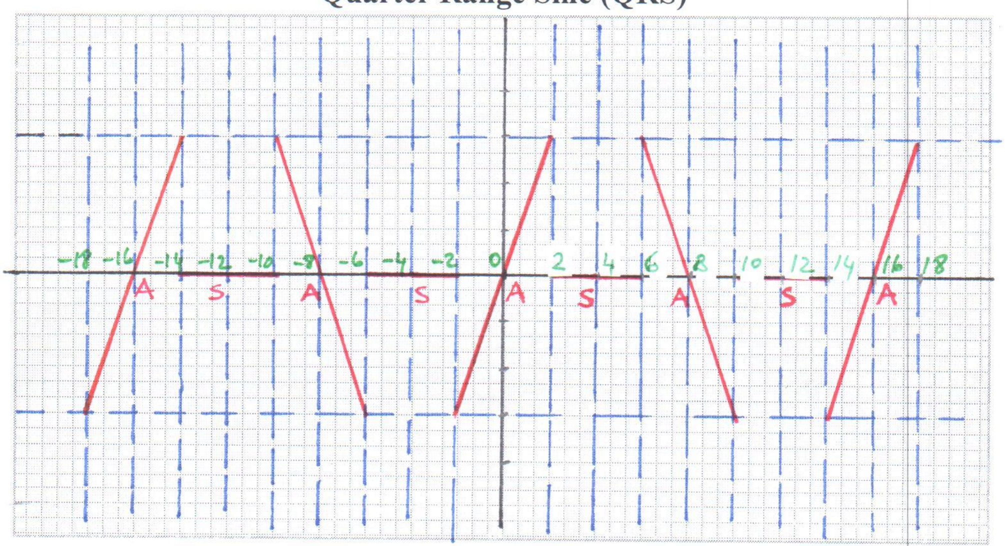
$$f(x) = \begin{cases} 3x & if\, 0 \le x < 2 \\ 0 & if\, 2 \le x < 4 \end{cases}$$

For the given function $f(x)$ prepare a labeled sketch of the quarter range sine (QRS) and quarter range cosine extension (QRC) of $f(x)$ and derive the corresponding expansions. The sketch should at least cover the interval $-18 < x < 18$

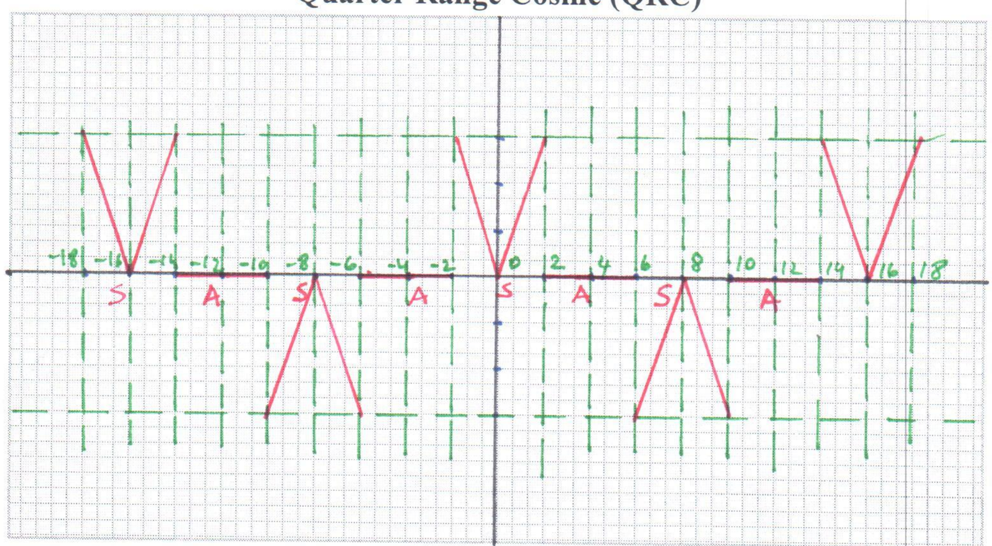
$$f(x) = \begin{cases} 3x & if\, 0 \le x < 2 \\ 0 & if\, 2 \le x < 4 \end{cases}$$

Cevap:

Quarter Range Sine (QRS)



Quarter Range Cosine (QRC)



Quarter-Range sine Expansion of $f(x)$

$$f(x) = \sum_{n=1,3,..}^{\infty} b_n \sin\frac{n\pi x}{2L} \qquad 0<x<L$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{2L}\,dx \quad ; \quad 0<x<4$$

Note that $L=4 \Rightarrow 2L=8$

$$b_n = \frac{2}{4}\int_0^4 f(x)\sin\frac{n\pi x}{8}\,dx = \frac{1}{2}\left[\int_0^2 (3x)\sin\frac{n\pi x}{8}\,dx + \underbrace{\int_2^4 (0)\sin\frac{n\pi x}{8}\,dx}_{0}\right]$$

$$= \frac{3}{2}\int_0^2 \underbrace{x}_{u}\,\underbrace{\sin\frac{n\pi x}{8}\,dx}_{dv} = \frac{3}{2}\left[x\left(-\frac{8}{n\pi}\right)\cos\frac{n\pi x}{8}\Big|_0^2 - \left(-\frac{8}{n\pi}\right)\int_0^2 \cos\frac{n\pi x}{8}\,dx\right]$$

$$= \frac{3}{2}\left[-\frac{16}{n\pi}\cos\frac{n\pi}{4} + \left(\frac{8}{n\pi}\right)^2 \sin\frac{n\pi x}{8}\Big|_0^2\right]$$

$$\boxed{b_n = -\frac{24}{n\pi}\cos\frac{n\pi}{4} + \frac{96}{n^2\pi^2}\sin\frac{n\pi}{4} \quad , \quad n=1,3,5,\ldots}$$

Quarter-Range cosine expansion of $f(x)$

$$f(x) = \sum_{n=1,3,..}^{\infty} a_n \cos\frac{n\pi x}{2L} \qquad 0<x<L$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{2L}\,dx \quad ; \quad 0<x<4$$

$$= \frac{2}{4}\int_0^4 f(x)\cos\frac{n\pi x}{8}\,dx = \frac{1}{2}\left[\int_0^2 (3x)\cos\frac{n\pi x}{8}\,dx + \underbrace{\int_2^4 (0).\cos\frac{n\pi x}{8}\,dx}_{0}\right]$$

$$a_n = \frac{3}{2}\int_0^2 \underbrace{x}_{u}\,\underbrace{\cos\frac{n\pi x}{8}\,dx}_{dv} = \frac{3}{2}\left[x\left(\frac{8}{n\pi}\right)\sin\frac{n\pi x}{8}\Big|_0^2 - \left(\frac{8}{n\pi}\right)\int_0^2 \sin\frac{n\pi x}{8}\,dx \right]$$

$$= \frac{3}{2}\left[\frac{16}{n\pi}\sin\frac{n\pi}{4} + \left(\frac{8}{n\pi}\right)^2 \cos\frac{n\pi x}{8}\Big|_0^2 \right]$$

$$\boxed{a_n = \frac{24}{n\pi}\sin\frac{n\pi}{4} + \frac{96}{n^2\pi^2}\cos\frac{n\pi}{4} - \frac{96}{n^2\pi^2}}$$

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Soru 3: (Heat Equation)

Bir çubuktaki sıcaklık dağılımının, $u(x,t)$ denklemi aşağıdaki gibidir

$$\alpha^2 u_{xx} - u_t = 0, \qquad (0 < x < L = \pi, \quad 0 < t < \infty)$$

$$u(0,t) = 20\ , \quad u_x(\pi,t) = 3, \quad (\text{i.e.,} \quad L = \pi)\ ; \ 0 < t < \infty$$

$$u(x,0) = 0\ ;\ 0 < x < \pi$$

Değişkenlerine ayırma yöntemini kullanarak $u(x,t)$ 'yi bulunuz.

Consider the temperature field $u(x,t)$ in a rod.

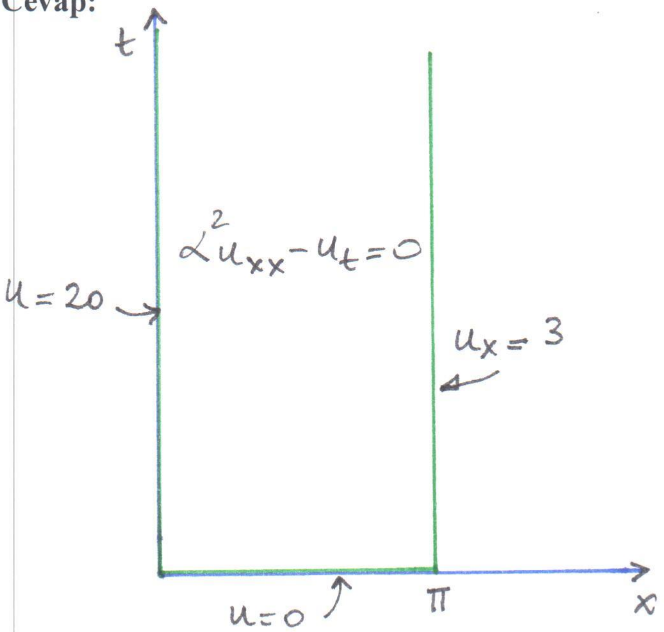
$$\alpha^2 u_{xx} - u_t = 0, \qquad (0 < x < L = \pi, \quad 0 < t < \infty)$$

$$u(0,t) = 20\ , \quad u_x(\pi,t) = 3, \quad (\text{i.e.,} \quad L = \pi)\ ; \ 0 < t < \infty$$

$$u(x,0) = 0\ ;\ 0 < x < \pi$$

Find the $u(x,t)$ using separation of variables

Cevap:



$$\alpha^2 u_{xx} - u_t = 0 \quad \text{———} \quad (1)$$

$$u(0,t) = 20 \quad \text{———} \quad (2)$$

$$u_x(\pi,t) = 3 \quad \text{———} \quad (3)$$

$$u(x,0) = 0 \quad \text{———} \quad (4)$$

$$u(x,t) = X(x)T(t) \quad \text{———} \quad (5)$$

substitute (5) into (1)

$$\alpha^2 X'' T - X T' = 0 \quad \text{———} \quad (6)$$

Or, dividing (6) by XT

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} \quad \text{———} \quad (7)$$

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -k^2 \quad \text{———} \quad (8)$$

$$X'' + k^2 X = 0 \quad \text{———} \quad (9a)$$

$$T' + k^2\alpha^2 T = 0 \quad \text{———} \quad (9b)$$

$$X = A\cos kx + B\sinh kx \quad \text{——— (10a)}$$

$$T = C e^{-k^2\alpha^2 t} \quad \text{——— (10b)}$$

$$X = \begin{cases} A\cos kx + B\sinh kx, & k \neq 0 \\ D + Ex, & k = 0 \end{cases} \quad \text{——— (11)}$$

$$T = \begin{cases} F e^{-k^2\alpha^2 t}, & k \neq 0 \\ G, & k = 0 \end{cases} \quad \text{——— (12)}$$

From (5)

$$u(x,t) = X(x)\,T(t)$$

$$= H + Ix + (J\cos kx + K\sinh kx)e^{-k^2\alpha^2 t} \quad \text{——— (13)}$$

where

$$H = D\cdot G \quad \text{——— (14)}$$

$$I = E\cdot G \quad \text{——— (15)}$$

$$J = A\cdot F \quad \text{——— (16)}$$

$$K = B\cdot F \quad \text{——— (17)}$$

Boundary condition at $x=0$ (Eq 2)

$$u(0,t) = 20 = H + J e^{-k^2\alpha^2 t} \quad \text{——— (18)}$$

suggests

$$H = 20 \quad \text{——— (19)}$$

$$J = 0 \quad \text{——— (20)}$$

Boundary condition at $x=\pi$ (Eq.3)

$$u_x(x,t) = I + (K.k.\cos kx)e^{-k^2\alpha^2 t} \quad \text{——— (21)}$$

$$u_x(\pi,t) = I + (K.k.\cos k\pi)\, e^{-k^2\alpha^2 t} = 3 \quad \text{——— (22)}$$

suggests

$$I = 3 \quad \text{——— (23)}$$

$$\cos k\pi = 0$$

$$k\pi = \frac{n\pi}{2}, \quad n = 1,3,5\ldots \quad \text{——— (24)}$$

$$k = \frac{n}{2}, \quad n = 1,3,5\ldots \quad \text{——— (25)}$$

$$u(x,t) = 20 + 3x + \sum_{1,3,5\ldots}^{\infty} L_n \sin\frac{nx}{2}\, e^{-\left(\frac{n\alpha}{2}\right)^2 t} \quad \text{——— (26)}$$

Apply initial condition at $t=0$

$$u(x,0) = 0 = 20 + 3x + \sum_{1,3,5}^{\infty} L_n \sin\frac{nx}{2} \quad \text{——— (27)}$$

Consider the Quarter-range sine expansion

$$f(x) = \sum_{n=1,3,\ldots}^{\infty} b_n \sin\frac{n\pi x}{2L} \qquad 0<x<L \quad \text{——— (28)}$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{2L}\,dx \quad \text{——— (29)}$$

if $L=\pi \Rightarrow$ $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x).\sin\frac{nx}{2}\,dx \quad \text{——— (30)}$$

Writing Eq. (27) in the following form

$$f(x) = -20 - 3x = \sum_{1,3,5}^{\infty} L_n \sin\frac{nx}{2} \quad \text{——— (31)}$$

Using (30) & (31)

$$L_n = \frac{2}{\pi}\int_0^{\pi} \underbrace{(-20-3x)}_{u} \underbrace{\sin\frac{nx}{2}\,dx}_{dv}$$

$$= \frac{2}{\pi}\left[(-20-3x)\left(-\frac{2}{n}\right)\cos\frac{nx}{2}\Big|_0^{\pi} - \int_0^{\pi}\left(-\frac{2}{n}\right)(-3)\cos\frac{nx}{2}\,dx\right]$$

$$= \frac{2}{\pi}\left[(-20-3\pi)\left(-\frac{2}{n}\right)\cos\frac{n\pi}{2} - (-20)\left(-\frac{2}{n}\right) - \frac{6}{n}\cdot\frac{2}{n}\sin\frac{nx}{2}\Big|_0^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{40+6\pi}{n}\right)\cos\frac{n\pi}{2} - \frac{40}{n} - \frac{12}{n^2}\sin\frac{n\pi}{2}\right]$$

$\therefore$

$$u(x,t) = 20 + 3x + \sum_{1,3,5}^{\infty} L_n \sin\frac{nx}{2}\, e^{-\left(\frac{n\alpha}{2}\right)^2 t}$$

For odd n $\quad \cos\frac{n\pi}{2} = 0$

$$L_n = -\frac{80}{\pi n} - \frac{24}{\pi n^2}\sin\frac{n\pi}{2}$$

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Soru 4: (Laplace Equation)

Değişkenlerine ayırma yöntemini kullanarak sınır değer problemini çözünüz.

(a) Değişkenlere ayırdıktan sonra bulunan sabitin pozitif, negatif ve sıfır olma durumlarını araştırınız.

İpucu: x'e bağlı çözümü $X(x)=Ce^{kx}+De^{-kx}$ varsayarak $u(x,y)$'ın sınırlı olma (boundedness) şartını kullanınız.

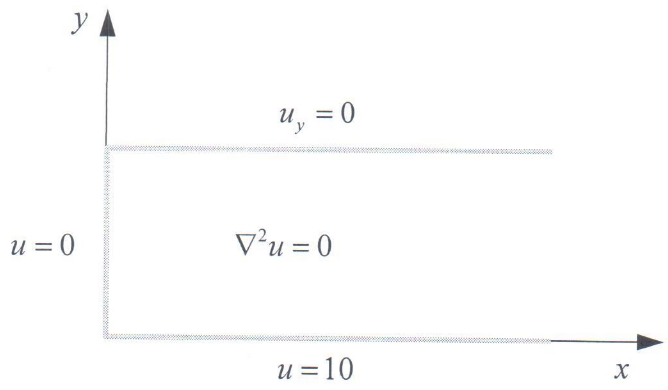
$$\nabla^2 u = u_{xx}+u_{yy}=0 \qquad (0<x<\infty,\quad 0<y<1)$$
$$u(x,0)=10,\quad u_y(x,1)=0,\quad (0<x<\infty)$$
$$u(0,y)=0,\quad (0<y<1)$$
$$u(x,y) \quad \text{bounded as} \quad x\to\infty$$

Solve the boundary value problem by using the method of seperation

(a) After seperating the variables, analyze the value of the constant for positive, negative and zero values.

$$\nabla^2 u = u_{xx}+u_{yy}=0 \qquad (0<x<\infty,\quad 0<y<1)$$
$$u(x,0)=10,\quad u_y(x,1)=0,\quad (0<x<\infty)$$
$$u(0,y)=0,\quad (0<y<1)$$
$$u(x,y) \quad \text{bounded as} \quad x\to\infty$$

Hint: Assume the solution of $X(x)=Ce^{kx}+De^{-kx}$ and use the boundedness condition of $u(x,y)$.



Cevap:

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MIDTERM SOLUTION

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \quad (0<x<\infty,\ 0<y<1) \qquad (1)$$

$$u(x,0) = 10 \quad (0<x<\infty) \qquad (2)$$

$$u_y(x,1) = 0 \quad (0<x<\infty) \qquad (3)$$

$$u(0,y) = 0 \quad (0<y<1) \qquad (4)$$

$u(x,y)$ bounded as $x \to \infty$

$$u(x,y) = X(x)\,Y(y) \qquad (5)$$

substitute (5) into (1)

$$X''Y + XY'' = 0 \qquad (6)$$

Dividing both sides of Eq. (6) by $X \cdot Y$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

or

$$\frac{X''}{X} = -\frac{Y''}{Y} = -k^2 \qquad (7)$$

$$X(x) = \begin{cases} A + Bx & k=0 \\ Ce^{kx} + De^{-kx} & k \neq 0 \end{cases} \qquad (8)$$

$$Y(y) = \begin{cases} E + Fy & k=0 \\ G\cos ky + H\sin ky & k \neq 0 \end{cases} \qquad (9)$$

$$u(x,y) = (A+Bx)(E+Fy) + (Ce^{kx} + De^{-kx})(G\cos ky + H\sin ky) \qquad (10)$$

Applying the boundedness condition ($u(x,y)$ bounded as $x \to \infty$) we see that $B=0$ and $C=0$.

Therefore Eq. (10) becomes

$$u(x,y) = I + Jy + (P\cos ky + Q\sin ky)e^{-kx} \quad \text{——} (11)$$

where

$$I = A \cdot E \quad \text{——} (12a)$$

$$J = A \cdot F \quad \text{——} (12b)$$

$$P = D \cdot G \quad \text{——} (12c)$$

$$Q = D \cdot H \quad \text{——} (12d)$$

Applying boundary condition at $y=0$ (Eq (2))

$$u(0,y) = 10 = I + Pe^{-kx} \quad \text{——} (13)$$

Matching the coefficients

$$I = 10 \quad \text{——} (14a)$$

$$P = 0 \quad \text{——} (14b)$$

Substituting (14a) & (14b) into (11)

$$u(x,y) = 10 + Jy + Q\sin ky\, e^{-kx} \quad \text{——} (15)$$

Applying the boundary condition at at $y=1$ (Eq 3)

$$u_y(x,y) = J + kQ\cos ky\, e^{-kx} \quad \text{——} (16)$$

$$u_y(x,1) = 0 = J + kQ\cos k\, e^{-kx} \quad \text{——} (17)$$

Eq (17) suggests that

$$J=0 \qquad \text{———} \quad (18a)$$

$$\cos k = 0 \Rightarrow k = n\frac{\pi}{2} \qquad n=1,3,5 \qquad \text{———} \quad (18b)$$

Substituting (18a) and (18b) into (15)

$$u(x,y) = 10 + \sum_{n=1,3,5}^{\infty} Q_n \sin\frac{n\pi y}{2} e^{-\frac{n\pi x}{2}} \qquad \text{———} \quad (19)$$

Applying the boundary condition at $x=0$

$$u(0,y) = 10 + \sum_{n=1,3,5..}^{\infty} Q_n \sin\frac{n\pi y}{2} = 0 \qquad \text{———} \quad (20)$$

Eq (19) can be rewritten in the following form

$$-10 = \sum_{n=1,3,5..}^{\infty} Q_n \sin\frac{n\pi y}{2} \qquad \text{———} \quad (21)$$

Half range sine expansion of (-10) $(0<y<1)$

$$Q_n = \frac{2}{1}\int_0^1 (-10)\cdot\sin\frac{n\pi y}{2}\,dy$$

$$= -20\left(-\frac{2}{n\pi}\right)\cos\frac{n\pi y}{2}\Big|_0^1$$

$$= \frac{40}{n\pi}\left(\cancelto{0}{\cos\frac{n\pi}{2}} - 1\right) = -\frac{40}{n\pi} \qquad \text{———} \quad (22)$$

$n=1,3,5...$

Substituting (22) into (19)

$$\boxed{u(x,y) = 10 - \frac{40}{\pi}\sum_{n=1,3,..}^{\infty} \frac{1}{n}\sin\frac{n\pi y}{2} e^{-\frac{n\pi x}{2}}}$$

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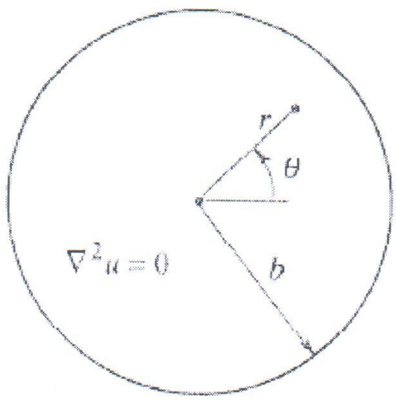
Soru 5: (Laplace Equation in polar coordinates)

1. Şekildeki Yuvarlak disk için Dirichlet problem verilmiştir. $u(r,\theta)$ bulunuz.

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad (0 \le r < b)\ ,\ 0 \le r < b$$
$$u(b,\theta) = f(\theta)\ ,\ -\infty < \theta < +\infty$$
$$f(\theta) = \begin{cases} 50 & ,0<\theta<\pi \\ 0 & ,\pi<\theta<2\pi \end{cases}$$
$u(r,\theta)$ bounded as $r \to 0$
$$u(r,\theta+2\pi) = u(r,\theta)$$

Consider the Dirichlet problem for circular disk. Find $u(r,\theta)$

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad (0 \le r < b)\ ,\ 0 \le r < b$$
$$u(b,\theta) = f(\theta)\ ,\ -\infty < \theta < +\infty$$
$$f(\theta) = \begin{cases} 50 & ,0<\theta<\pi \\ 0 & ,\pi<\theta<2\pi \end{cases}$$
$u(r,\theta)$ bounded as $r \to 0$
$$u(r,\theta+2\pi) = u(r,\theta)$$



Cevap:

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad \text{——— (1)}$$

$$u(b,\theta) = f(\theta), \quad -\infty < \theta < \infty \qquad \text{——— (2)}$$

$$f(\theta) = \begin{cases} 50, & 0 < \theta < \pi \\ 0, & \pi < \theta < 2\pi \end{cases} \qquad \text{——— (3)}$$

$u(r,\theta)$ bounded as $r \to 0$ ——— (4)

$$u(r,\theta+2\pi) = u(r,\theta) \qquad \text{——— (5)}$$

Seperating the variables

$$u(r,\theta) = R(r)\,\Theta(\theta) \qquad \text{——— (6)}$$

Substituting (6) into (1)

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \qquad \text{——— (7)}$$

Multiplying both sides of Eq (7) by $\frac{r^2}{R\Theta}$

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \text{constant} = \mu^2 \qquad \text{——— (8)}$$

$$r^2R'' + rR' - \mu^2 R = 0 \qquad \text{——— (9)}$$

$$\Theta'' + \mu^2\Theta = 0 \qquad \text{——— (10)}$$

Assuming the solution of (9) of the form $R = r^\lambda$ and substitute into (9)

$$r^2\lambda(\lambda-1)r^{\lambda-2} + r\lambda r^{\lambda-1} - \mu^2 r^\lambda = 0$$

$$(\lambda(\lambda-1) + \lambda - \mu^2)r^\lambda = 0 \Rightarrow \lambda^2 - \mu^2 = 0 \quad \text{— (11)}$$

$\lambda = \mp\mu$ —————————— (12)

Two linearly independent solution of (9) are

$R = r^{\mu}$ and $R = r^{-\mu}$ if $\mu \neq 0$ ———— (13)

If $\mu = 0$ from Eq (9) we have

$$r^2 R'' + rR' = 0 \quad \text{———————— (14)}$$

Let $R' = p \Rightarrow R'' = \dfrac{dp}{dr}$ ———— (15)

Substitute (15) into (14)

$$r^2 \frac{dp}{dr} + rp = 0$$

or dividing by r

$$r\frac{dp}{dr} + p = 0$$

or

$$r\,dp + p\,dr = 0$$

$$d(rp) = 0$$

$$rp = c_1 \Rightarrow p = \frac{c_1}{r}$$

From (15)

$$R(r) = \int p\,dr = \int \frac{c_1}{r}\,dr = c_1 \ln r + c_2$$

$$R(r) = \begin{cases} A + B\ln r & \mu = 0 \\ Cr^{\mu} + Dr^{-\mu} & \mu \neq 0 \end{cases}$$

$$\Theta(\theta) = \begin{cases} E + F\theta & \mu = 0 \\ G\cos\mu\theta + H\sin\mu\theta & \mu \neq 0 \end{cases}$$

MIDTERM SOLUTION

$$u(r,\theta) = (A + B\ln r)(E + F\theta) + (Cr^{\mu} + Dr^{-\mu})(G\cos\mu\theta + H\sin\mu\theta)$$

Boundedness condition at $r=0$

$u(r,\theta)$ bounded as $r \to 0$

As $r \to 0$ $\quad \ln r \to -\infty \quad \Rightarrow B = 0$

As $r \to 0$ $\quad r^{-\mu} \to \infty \quad \Rightarrow D = 0$

$$u(r,\theta) = I + J\theta + r^{\mu}(P\cos\mu\theta + Q\sin\mu\theta)$$

where

$I = A \cdot E$

$J = A \cdot F$

$P = C \cdot G$

$Q = C \cdot H$

Single valuedness condition in θ

$$u(r, \theta + 2\pi) = u(r,\theta)$$

$\therefore$

$$\underbrace{\cos\mu(\theta+2\pi)} = \cos\mu\theta$$

$$\cos\mu\theta\cos 2\pi\mu - \sin\mu\theta \cdot \sin 2\pi\mu = \cos\mu\theta$$

Equating right and left sides

$\cos 2\pi\mu = 1 \; ; \Rightarrow \mu = 1, 2, 3 \ldots$

$\sin 2\pi\mu = 0 \quad \Rightarrow \mu = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

Therefore $\mu = 1, 2, 3, \ldots \Rightarrow \mu = n \;, \; n = 1, 2, 3 \ldots$

$J\theta$ is not periodic

Therefore $J=0$

$$u(r,\theta) = I + \sum_{n=1}^{\infty} r^n \left(P_n \cos n\theta + Q_n \sin n\theta \right)$$

Boundary condition at $r=b$

$$u(b,\theta) = f(\theta) = I + \sum_{n=1}^{\infty} b^n \left(P_n \cos n\theta + Q_n \sin n\theta \right)$$

This is full Fourier series expansion of $f(\theta)$

$$I = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta = \frac{1}{2\pi}\int_{0}^{2\pi} f(\theta)\,d\theta = \frac{1}{2\pi}\int_{0}^{\pi} 50\,d\theta = \frac{1}{2\pi}\cdot 50\pi = \frac{50}{2}$$

$$P_n = \frac{1}{\pi b^n}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta = \frac{1}{\pi b^n}\int_{0}^{2\pi} f(\theta)\cos n\theta\,d\theta$$

$$Q_n = \frac{1}{\pi b^n}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta = \frac{1}{\pi b^n}\int_{0}^{2\pi} f(\theta)\sin n\theta\,d\theta$$

$$P_n = \frac{1}{\pi}\int_{0}^{\pi} 50\cos n\theta\,d\theta = \frac{50}{\pi}\,\frac{1}{n}\sin n\theta\Big|_0^{\pi} = 0$$

$$Q_n = \frac{1}{\pi b^n}\int_{0}^{\pi} 50\sin n\theta = -\frac{50}{\pi}\,\frac{1}{n}\,\frac{\cos n\theta}{b^n}\Big|_0^{\pi} = -\frac{50}{b^n \pi n}\left(\cos n\pi - 1\right)$$

$$Q_n = \frac{50}{b^n \pi n}\left(1-(-1)^n\right)$$

$\therefore$

$$u(r,\theta) = \frac{50}{2} + \frac{50}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{b}\right)^n\cdot\left(1-(-1)^n\right)\sin n\theta$$