MAK501 Advanced Engineering Math (Fall 2016)

Midterm 2 Examination (Vize 2 Sınavı)

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Question No	Maximum Point	Point
1	30	29
2	30	30
3	30	30
4	30	
Total	120	

Instructions

1. Yükseköğretim Kurumları 2015 Öğrenci Disiplin Yönetmeliği Madde 5-d ve 7-e'ye göre "sınavlarda kopyaya teşebbüs veya kopya çekmek yapmak veya yaptırmak veya bunlara teşebbüs etmek" fiilinin suçu YÜKSEKÖĞRETİM KURUMUNDAN BİR VEYA İKİ YARIYIL İÇİN UZAKLAŞTIRMA cezasıdır.

UYARI VE KURALLARI OKUDUM.

Signature: [signature]

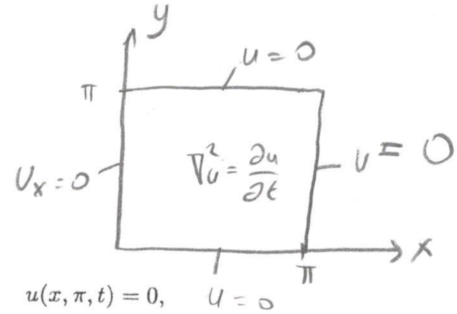
Good luck!

3 tane fazladan sayfa vardır

Question 1 (30 points)

Solve the heat equation in a rectangle $0 < x < \pi$, $0 < y < \pi$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$



subjected to the boundary conditions

$$u_x(0, y, t) = 0, \quad u(\pi, y, t) = 0, \quad u(x, 0, t) = 0, \quad u(x, \pi, t) = 0,$$

and the initial condition

$$u(x, y, 0) = (\sin 2y + \sin 3y)\cos\frac{x}{2}.$$

$u = XYT$ → separation of variables

$$\left[XYT' = X''YT + XY''T\right]\frac{1}{XYT}$$

$$\frac{T'}{T} = \frac{X''}{X} + \frac{Y''}{Y} = -k^2 = \text{constant}$$

$$\frac{X''}{X} = -\frac{Y''}{Y} - k^2 = -p^2 = \text{constant}$$

$T' = -k^2 T$

$X'' = -p^2 X$

$Y'' = -(k^2 - p^2)Y = -s^2 Y, \quad s^2 = k^2 - p^2$

k=0 durumuna bakmamıza gerek yoktur.

$T' = -k^2 T \Rightarrow r = -k^2 \Rightarrow T = C_1 e^{-k^2 t}$

$X'' = -p^2 X \Rightarrow r^2 = -p^2, \; r = \pm ip \Rightarrow X = C_2\cos(px) + C_3\sin(px)$

$Y'' = -s^2 Y \Rightarrow r^2 = -s^2, \; r = \pm is \Rightarrow Y = C_4\cos(sy) + C_5\sin(sy)$

$u = XYT = (C_2\cos(px) + C_3\sin(px))(C_4\cos(sy) + C_5\sin(sy))C_1 e^{-k^2 t}$

$u(x,y,t) = (A\cos(px) + B\sin(px))(C\cos(sy) + D\sin(sy))e^{-k^2 t}$

$A = C_2 C_1, \quad B = C_3 C_1, \quad C = C_4 C_1, \quad D = C_5 C_1$

ekstra: $r = \pm ir$

$C_1 e^{ir} + C_2 e^{-ir} = C_1(\cos r + i\sin r) + C_2(\cos r - i\sin r)$

$= \underbrace{(C_1 + C_2)}_{A}\cos r + \underbrace{(C_1 - C_2)i}_{B}\sin r$

$$u(x,y,t) = (A\cos(px) + B\sin(px))(C\cos(sy) + D\sin(sy))\, e^{-k^2 t}$$

<u>Boundary Cond's</u>

$$u_x(x,y,t) = (-A\sin(px)\,p + B\cos(px)p)(C\cos(sy) + D\sin(sy))\, e^{-k^2 t}$$

$$\rightarrow u_x(0,y,t) = (-A\cancel{\sin(0)}^{0} p + B\cancel{\cos(0)}^{1} p)(C\cos(sy) + D\sin(sy))\, e^{-k^2 t} = 0$$

$$B = 0$$

$$u(x,y,t) = A\cos(px)\cdot(C\cos(sy) + D\sin(sy))\, e^{-k^2 t}$$

$$\rightarrow u(\pi,y,t) = A\cos(p\pi)\cdot(C\cos(sy) + D\sin(sy))\, e^{-k^2 t} = 0$$

$$A \neq 0,\ C \neq 0,\ D \neq 0,\ \cos(p\pi) = 0$$

$$p\pi = n\frac{\pi}{2},\ n = 1,3,5,7$$

$$\boxed{p = \frac{n}{2}}$$

$$u(x,y,t) = \sum_{n=1,3,5}^{\infty} \cos\left(\frac{nx}{2}\right)\left(E_n\cos(sy) + F_n\sin(sy)\right) e^{-k^2 t}, \quad E_n = A_n\cdot C_n,\ F_n = A_n\cdot D_n$$

$$\rightarrow u(x,0,t) = \sum_{n=1,3,5}^{\infty} \cos\left(\frac{nx}{2}\right)\left(E_n\cancel{\cos(0)}^{1} + F_n\cancel{\sin(0)}^{0}\right) e^{-k^2 t} = 0$$

$$E_n = 0$$

$$u(x,y,t) = \sum_{n=1,3,5}^{\infty} \cos\left(\frac{nx}{2}\right) F_n \sin(sy)\, e^{-k^2 t}$$

$$\rightarrow u(x,\pi,t) = \sum_{n=1,3,5}^{\infty} \cos\left(\frac{nx}{2}\right) F_n\cdot\sin(s\pi)\, e^{-k^2 t} = 0$$

$$F_n \neq 0,\ \sin(s\pi) = 0$$

$$s\pi = m\pi,\ m = 1,2,3,4$$

$$s = m \Rightarrow k^2 = s^2 + p^2 = (n/2)^2 + (m)^2 = \frac{n^4}{4} + m^2$$

$$u(x,y,t) = \sum_{n=1,3,5}^{\infty}\sum_{m=1}^{\infty} F_{nm}\cos\left(\frac{nx}{2}\right)\sin(my)\, e^{-\left(\frac{n^4}{4} + m^2\right)t}$$

$$u(x,y,0) = \sum_{n=1,3,5}^{\infty}\sum_{m=1}^{\infty} F_{nm}\cos\left(\frac{nx}{2}\right)\sin(my) = f(x,y)$$

$y \rightarrow$ half-range sine
$x \rightarrow$ quarter-range cosine

$$F_{nm} = \frac{2}{L_x}\cdot\frac{2}{L_y}\int_0^{L_x}\int_0^{L_y} f(x,y)\cdot\cos\left(\frac{nx}{2}\right)\sin(my)\, dx\, dy$$

$$L_x = \pi,\ L_y = \pi,\ f(x,y) = (\sin 2y + \sin 3y)\cos\frac{x}{2}$$

Devamı diğer kağıtta

1'in devamı

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$$F_{nm} = \frac{\cancel{2}4}{\pi^2}\int_0^{\pi}(\sin 2y + \sin 3y)\sin my \cdot dy \int_0^{\pi} \cos\frac{x}{2}\cos\frac{nx}{2}$$

$2y = my$ $\quad$ $3y = my$ $\quad$ $\frac{x}{2} = \frac{nx}{2}$

$m=2$ $\quad$ $m = 3$ $\quad$ $n = 1$ olmalı

$$F_{1,2} = \frac{4}{\pi^2}\int_0^{\pi} \sin^2 2y \cdot dy \int_0^{\pi} \cos^2\frac{x}{2} \cdot dx$$

$$F_{1,2} = \frac{\cancel{4}}{\cancel{\pi^2}} \cdot \frac{1}{\cancel{2}}\cancel{\pi} \cdot \frac{1}{\cancel{2}}\cancel{\pi} = 1$$

$$F_{1,3} = \frac{4}{\pi^2}\int_0^{\pi} \sin^2 3y\, dy \cdot \int_0^{\pi} \cos^2\frac{x}{2} \cdot dx$$

$$F_{1,3} = \frac{4}{\pi^2} \cdot \frac{1}{2}\pi \cdot \frac{1}{2}\pi = 1$$

$$u(x,y,t) = \sum_{n=1}^{1}\sum_{m=1}^{2} \underbrace{F_{nm}}_{\text{her zaman } 1} \cos\left(\frac{nx}{2}\right)\sin(my)\, e^{-\left(\frac{n^4}{4}+m^2\right)t}$$

$$u(x,y,t) = \sum_{m=1}^{2} \cos\left(\frac{x}{2}\right)\sin(my)\, e^{-\left(\frac{1}{4}+m^2\right)t}$$

3

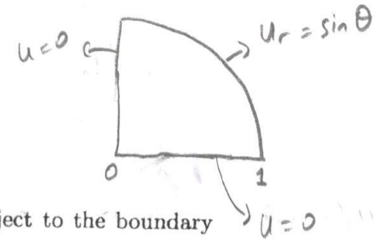
m=2

-1

$$= \cos\frac{x}{2}\sin 2y\, e^{-\frac{17}{4}t} + \cos\frac{x}{2}\sin 3y\, e^{-\frac{37}{4}t}$$

Question 2 (30 points)

Solve Laplace's equation



$$0 = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \quad (0 < r < 1,\ 0 < \theta < \pi/2),$$

inside a quarter-circle $0 < r < 1$, $0 < \theta < \pi/2$ (in polar coordinates r, θ) subject to the boundary conditions

$$u(r,0) = 0, \quad u(r,\pi/2) = 0, \quad |u(0,\theta)| < \infty, \quad u_r(1,\theta) = \sin\theta. = f(\theta)$$

$u = R\Theta$ separation of variables

$$\left[R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}\cdot R\Theta'' = 0\right]\cdot\frac{1}{R\Theta}$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\cdot\frac{\Theta''}{\Theta} = 0$$

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0 \Rightarrow \frac{r^2R''+rR'}{R} = -\frac{\Theta''}{\Theta} = k^2 = \text{constant}$$

$$r^2R''+rR'-k^2R=0 \quad , \quad \Theta'' = -k^2\Theta$$

$R = r^\lambda$

$\cancel{r^2}\,\lambda(\lambda-1)\cancel{r^{\lambda-2}} + \cancel{r}\,\lambda\,\cancel{r^{\lambda-1}} - k^2\cancel{r^\lambda} = 0$

$\lambda^2 - \cancel{\lambda} + \cancel{\lambda} - k^2 = 0$

$\lambda^2 = k^2$

$\lambda = \pm k$

$R = C_1 r^k + C_2 r^{-k}, \quad k \neq 0$

$r^2R'' + rR' = 0$

$R'' + \frac{1}{r}R' = 0$

$R' = P$

$P' + \frac{1}{r}P = 0$

$P = C_3\frac{1}{r} = R'$

$R' = C_3\frac{1}{r}$

$R = C_3\ln r + C_4, \quad k = 0$

$r^2 = -k^2, \quad r = \pm ik$

$\Theta = C_5\cos k\theta + C_6\sin k\theta, \quad k \neq 0$

$\Theta'' = 0$

$\Theta = C_7\theta + C_8, \quad k = 0$

$$R = \begin{cases} C_1 r^k + C_2 r^{-k} \\ C_3\ln r + C_4 \end{cases}$$

$$\Theta = \begin{cases} C_5\cos k\theta + C_6\sin k\theta \\ C_7\theta + C_8 \end{cases}$$

$u = (R\,\Theta)_1 + (R\,\Theta)_2$

$u(r,\theta) = (\cancel{C_3 \ln r} + C_4)(C_7\theta + C_8) + (C_1 r^k + \cancel{C_2 r^{-k}})(C_5 \cos k\theta + C_6 \sin k\theta)$

BC's

$\rightarrow \quad |u(0,\theta)| < \infty$

$r \rightarrow 0$

$C_3 \ln r$ olamaz $\Rightarrow C_3 = 0$

$C_2 r^{-k}$ olamaz $\Rightarrow C_2 = 0$

$u(r,\theta) = A\theta + B + r^k(C\cos k\theta + D \sin k\theta)$

$\rightarrow u(r,0) = B + r^k(C\underbrace{\cos(0)}_{1} + D\underbrace{\sin(0)}_{0}) = 0$

$B = 0, \quad C = 0$

$u(r,\theta) = A\theta + r^k D \sin k\theta$

$\rightarrow u(r,\pi/2) = A\frac{\pi}{2} + r^k D \sin k\frac{\pi}{2} = 0$

$A = 0, \quad D \neq 0, \quad \sin\frac{k\pi}{2} = 0$

$\frac{k\cancel{\pi}}{2} = n\cancel{\pi}, \quad n = 1, 2, 3, 4 \ldots$

$\boxed{k = 2n}$

$u(r,\theta) = \sum_{n=1}^{\infty} D_n \cdot r^{2n} \sin(2n\theta)$

$\rightarrow \quad u_r(r,\theta) = \sum_{n=1}^{\infty} D_n . 2n . r^{(2n-1)} \sin(2n\theta)$

$u_r(1,\theta) = \sum_{n=1}^{\infty} D_n . 2n . (1^{2n-1}) . \sin(2n\theta) = \sin\theta$

$\cancel{2}n . D_n = \frac{\cancel{2}}{\pi/2} \int_0^{\pi/2} \sin\theta . \sin(2n\theta) . d\theta$

$D_n = \frac{2}{n\pi} \int_0^{\pi/2} \sin\theta . \sin(2n\theta) . d\theta$

Devamı diğer sayfada

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2'nin devamı

$$D_n = \frac{2}{n\pi}\int_0^{\pi/2} \sin\theta \ \sin(2n\theta).d\theta$$

$$\sin a \sin b = \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right)$$

$$D_n = \frac{\cancel{2}}{n\pi}\int_0^{\pi/2} \frac{1}{\cancel{2}}\cdot\left(\cos(\theta - 2n\theta) - \cos(\theta + 2n\theta)\right) d\theta$$

$$D_n = \frac{1}{n\pi}\int_0^{\pi/2} \left(\cos(2n-1).\theta - \cos(2n+1)\theta\right).d\theta$$

$$D_n = \frac{1}{n\pi}\left[\frac{1}{2n-1}\sin(2n-1)\theta - \frac{1}{2n+1}\sin(2n+1)\theta\right]_0^{\pi/2}$$

$$D_n = \frac{1}{n\pi}\left[\frac{1}{2n-1}\left(\sin\frac{(2n-1)\pi}{2} - \cancel{\sin 0}^{\,0}\right) - \frac{1}{2n+1}\left(\sin\frac{(2n+1)\pi}{2} - \cancel{\sin 0}^{\,0}\right)\right]$$

$$D_n = \frac{1}{n\pi}\left(\frac{1}{2n-1}\left(\sin\left(\frac{(2n-1)\pi}{2}\right)\right) - \frac{1}{2n+1}\left(\sin\left(\frac{(2n+1)\pi}{2}\right)\right)\right)$$

1 1

$$u(r,\theta) = \sum_{n=1}^{\infty} D_n \ r^{2n} \sin(2n\theta)$$

$$D_n = \frac{1}{n\pi}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right] = \frac{1}{n\pi}\ \frac{2n+1-2n+1}{(4n^2-1)}$$

$$= \frac{2}{n\pi(4n^2-1)}$$

Question 3 (30 points) Solve the initial value problem for the heat equation on the infinite interval

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (-\infty < x < \infty,\ t > 0),$$

subjected to the boundary condition

$$u(x,0) = e^{-x^2/2}.$$

Domain $\Rightarrow -\infty < x < \infty \Rightarrow$ Fourier Transform

$$F\left(\frac{\partial u}{\partial t}\right) = \frac{d}{dt}\hat{u}, \quad F\left(\frac{\partial^2 u}{\partial x^2}\right) = (iw)^2 \hat{u} = -w^2 \hat{u}, \quad \hat{u} = \hat{u}(w,t)$$

$$F(u(x,0)) = \hat{u}(w,0) = F(e^{-x^2/2}) = \frac{1}{\sqrt{2 \cdot \frac{1}{2}}} \cdot e^{-\frac{w^2}{2}} = e^{-w^2/2}$$

$$\cancel{+}a\cancel{x^2} = \cancel{+}\frac{\cancel{x^2}}{2} \Rightarrow a = \frac{1}{2}$$

$$\frac{d}{dt}\hat{u} = -w^2 \hat{u}$$

$$r = -w^2 \Rightarrow \hat{u} = A(w) . e^{-w^2 t}$$

$$\hat{u}(w,0) = A(w) \cancel{e^{0}}^{1} = e^{-w^2/2}$$

$$\hat{u} = e^{-w^2/2} . e^{-w^2 t} = e^{-w^2(t+\frac{1}{2})}$$

$$F^{-1}\left(e^{-w^2(t+\frac{1}{2})}\right) = \sqrt{2 \cdot \frac{1}{(4t+2)}} \cdot e^{-\left(\frac{1}{4t+2}\right) \cdot x^2}$$

$$\cancel{+}\frac{\cancel{w^2}}{4b} = \cancel{+}\cancel{w^2}(t+\tfrac{1}{2}), \quad 4b = \frac{1}{t+\frac{1}{2}} \Rightarrow b = \frac{1}{4t+2}, \quad b > 0$$

$$F^{-1}(\hat{u}) = \underbrace{u(x,t) = \frac{1}{\sqrt{2t+1}} \cdot e^{-\frac{1}{2}\left(\frac{1}{2t+1}\right)x^2}}$$