MAK501 Advanced Engineering Math (Fall 2016)

Midterm 2 Examination (Vize 2 Sınavı)

November 22, 2016, 18:30 - 20:30

Name: ____________________

ID Number: ____________________

Question No	Maximum Point	Point
1	30	
2	30	
3	30	
4	30	
Total	120	

Instructions

1. Yükseköğretim Kurumları 2015 Öğrenci Disiplin Yönetmeliği Madde 5-d ve 7-e'ye göre "sınavlarda kopyaya teşebbüs veya kopya çekmek yapmak veya yaptırmak veya bunlara teşebbüs etmek" fiilinin suçu YÜKSEKÖĞRETİM KURUMUNDAN BİR VEYA İKİ YARIYIL İÇİN UZAKLAŞTIRMA cezasıdır.

UYARI VE KURALLARI OKUDUM.

Signature: ____________________

Good luck!

Question 1 (30 points)

Solve the heat equation in a rectangle $0 < x < \pi,\ 0 < y < \pi$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

subjected to the boundary conditions

$$u_x(0, y, t) = 0, \quad u(\pi, y, t) = 0, \quad u(x, 0, t) = 0, \quad u(x, \pi, t) = 0,$$

and the initial condition

$$u(x, y, 0) = (\sin 2y + \sin 3y) \cos \frac{x}{2}.$$

$$u(x,y,t) = \phi(x)\, h(y)\, G(t)$$

$$\frac{G'(t)}{G(t)} = \frac{\phi''(x)}{\phi(x)} + \frac{h''(y)}{h(y)} = -(\lambda^2 + \mu^2)$$

$$\frac{\phi''(x)}{\phi(x)} = -\lambda^2 \qquad \frac{h''(y)}{h(y)} = -\mu^2$$

$$\frac{\phi''(x)}{\phi(x)} = -\lambda$$

$$\Rightarrow$$

$$\phi_x(0)\, h(y)\, G(t) = \phi(\pi)\, h(y)\, G(t) = 0$$

$$\phi(x)\, h(0)\, G(t) = \phi(x)\, h(\pi)\, G(t) = 0$$

$$\phi_x(0) = 0, \quad \phi(\pi) = 0$$

$$h(0) = 0 \qquad h(\pi) = 0$$

$$\phi'' = -\lambda^2 \phi \qquad \phi_x(0) = \phi(\pi) = 0$$

$$\phi(x) = A\cos\lambda x + B\sin\lambda x$$

$$\phi'(x) = -A\sin\lambda x + B\lambda\cos\lambda x$$

$$\phi_x(0) = B\lambda = 0 \Rightarrow B = 0$$

$$\phi(\pi) = A\cos\lambda\pi = 0 \qquad \lambda = \frac{n}{2}, \quad n = 1, 3, 5 -$$

$$\phi(x) = \cos\frac{nx}{2}$$

$$\frac{h''}{h} = -\mu^2$$

$$h(y) = C\cos\mu y + D\sin\mu y$$

$$h(0) = C = 0$$

$$h(\pi) = D\sin\mu\pi = 0 \qquad \mu\pi = m\pi$$

$$\mu = m$$

$$h(y) = \sin my$$

$$\frac{G'}{G} = -(\lambda^2 + \mu^2) = -(n^2 + m^2)$$

$$G = E e^{-\left(\frac{n^2}{4} + m^2\right)t}$$

$$u_{nm}(x,y,t) = e^{-\left(\frac{n^2}{4} + m^2\right)t} \cos\frac{nx}{2} \sin my \qquad n = 1, 3, 5$$

$$m = 1, 2, \ldots$$

$$u(x,y,t) = \sum_{n=1,3..}^{\infty} \sum_{m=1}^{\infty} C_{nm} e^{-\left(\frac{n^2}{4} + m^2\right)t} \cos\frac{nx}{2} \sin my$$

$$u(x,y,0) = (\sin 2y + \sin 3y)\cos\frac{x}{2}$$

$$(\sin 2y + \sin 3y)\cos\frac{x}{2} = \sum_{n=1,3..} \sum_{m=1} C_{nm} \cos\frac{nx}{2} \sin my$$

$$n = 1,\ m = 2 \qquad C_{12} = 1 \qquad \frac{1}{4} + 4 = \frac{17}{4}$$

$$n = 1,\ m = 3 \qquad C_{13} = 1 \qquad \frac{1}{4} + 9 = \frac{37}{4}$$

$$u(x,y,t) = e^{-\left(\frac{17}{4}t\right)} \cos\frac{x}{2} \sin 2y + e^{-\left(\frac{37}{4}t\right)} \cos\frac{x}{2} \sin 3y$$

Question 2 (30 points)

Solve Laplace's equation

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \quad (0<r<1,\ 0<\theta<\pi/2),$$

inside a quarter-circle $0<r<1$, $0<\theta<\pi/2$ (in polar coordinates r,θ) subject to the boundary conditions

$$u(r,0)=0, \quad u(r,\pi/2)=0, \quad |u(0,\theta)|<\infty, \quad u_r(1,\theta)=\sin\theta.$$

$u(r,\theta) = h(r)\,\phi(\theta)$

$h''(r)\phi(\theta) + \frac{1}{r}h'(r)\phi(\theta) + \frac{1}{r^2}h(r)\phi''(\theta) = 0,$

$$\frac{r^2h''(r)+rh'(r)}{h(r)} = -\frac{\phi''(\theta)}{\phi(\theta)} = \lambda^2$$

$r^2h''(r)+rh'(r)=\lambda^2 h(r)$

$\phi'' = -\lambda^2\phi$

$h(r)\phi(0)=0, \Rightarrow \phi(0)=0$

$h(r)\phi(\pi/2)=0 \Rightarrow \phi(\pi/2)=0$

$|h(0)\phi(\theta)|<\infty \Rightarrow |h(0)|<\infty$

$\phi = A\cos\lambda_n\theta + B\sin\lambda_n\theta$ $\quad$ $\phi(0)=0=A=0$

$\phi(\pi/2)=B\sin\frac{\lambda\pi}{2}=0$

$\lambda\frac{\pi}{2}=n\pi$

$\lambda=2n$

$h(r)=c_1r^\mu+\cancel{c_2r^{-\mu}}^{\to 0}$

$r^2\mu(\mu-1)r^{\mu-2}+r\mu r^{\mu-1}=\lambda^2r^\mu$

$\mu^2-\cancel{\mu}+\cancel{\mu}=\lambda^2$

$\mu=\lambda$

$h(r)=\sum_{n=1}^{\infty} c_n r^{2n}\sin 2n\theta$

Question 3 (30 points) Solve the initial value problem for the heat equation on the infinite interval

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (-\infty < x < \infty,\ t > 0),$$

subjected to the boundary condition

$$u(x,0) = e^{-x^2/2}.$$

$$F\left(\frac{\partial u}{\partial t}\right) = \frac{d\hat{u}}{dt}$$

$$F\left(\frac{\partial^2 u}{\partial x^2}\right) = (i\omega)^2 \hat{u} = -\omega^2 \hat{u}$$

$$u(x,t) \xrightarrow{\text{Fourier}} \hat{u}(\omega,t)$$

$$\frac{d\hat{u}}{dt} = -\omega^2 \hat{u}$$

$$\hat{u} = A(\omega)\, e^{-\omega^2 t}$$

I.C. $\hat{u}(\omega,0) = A(\omega) = F\left(e^{-\frac{x^2}{2}}\right)$

$$F\left(e^{-\frac{a}{2}x^2}\right) = \frac{1}{\sqrt{a}}\, e^{-\frac{\omega^2}{2a}}$$

$$a=1 \Rightarrow F\left(e^{-\frac{x^2}{2}}\right) = e^{-\frac{\omega^2}{2}}$$

$$\hat{u} = A(\omega)\, e^{-\omega^2 t} = e^{-\frac{\omega^2}{2}} \cdot e^{-\omega^2 t} = e^{-\omega^2\left(\frac{1}{2}+t\right)}$$

$$u(x,t) = F^{-1}\left(\hat{u}(\omega,t)\right) = F^{-1}\left(e^{-\omega^2\left(\frac{1}{2}+t\right)}\right)$$

$$F^{-1}\left(e^{-\frac{\omega^2}{2a}}\right) = \sqrt{a}\, e^{-\frac{a}{2}x^2} \quad ; \quad \text{Let } 2a = \frac{1}{\frac{1}{2}+t} \ ; \quad a = \frac{1}{1+2t}$$

$$F^{-1}\left(e^{-\omega^2\left(\frac{1}{2}+t\right)}\right) = \sqrt{\frac{1}{1+2t}} \cdot e^{-\frac{x^2}{2+4t}}$$

$$u(x,t) = \frac{1}{\sqrt{1+2t}}\, e^{-\frac{x^2}{2+4t}}$$

Question 4 (30 points) Solve the heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + \sin 2\pi x, \quad 0 < x < 1, \quad 0 < t < \infty,$$

subjected to the boundary conditions

$$u(0,t) = 0, \quad 0 < t < \infty$$
$$u(1,t) = 0, \quad 0 < t < \infty$$

and the initial condition

$$u(x,0) = \sin \pi x \quad 0 \le x \le 1.$$

Solution of the homogeneous problem

$$u_t = \alpha^2 u_{xx}$$

$$u(x,t) = X(x)\,T(t)$$

$$X(0) = 0$$

$$X(1) = 0$$

$$\frac{1}{\alpha^2}\frac{T'}{T} = \frac{X''}{X} = -\lambda^2$$

$$X(x) = A\cos\lambda x + B\sin\lambda x$$

$$X(0) = A = 0$$

$$X(1) = B\sin\lambda = 0 \quad \Rightarrow \quad \lambda = n\pi$$

$$X_n(x) = \sum_{n=1}^{\infty} B_n \sin\lambda_n x = \sum_{n=1}^{\infty} B_n \sin n\pi x$$

expanding the nonhomogeneous term ($\sin 2\pi x$) into

$$\sin 2\pi x = \sum_{n=1}^{\infty} C_n^{(t)} \sin n\pi x$$

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t) = \sum_{n=1}^{\infty} T_n(t) B_n \sin n\pi x$$

Substitute

$$\sum_{n=1}^{\infty} T_n'(t) B_n \sin n\pi x = \alpha^2 \sum_{n=1}^{\infty} (n\pi)^2 B_n \sin n\pi x\, T_n(t) + \sum_{n=1}^{\infty} C_n \sin n\pi x$$

$$\sum_{n=1}^{\infty} \left[T_n'(t) + (n\pi\alpha)^2 T_n(t) - C_n(t) \right] B_n \sin n\pi x = 0$$

$$\sum_{n=1}^{\infty} T_n(0) \sin n\pi x \, B_n = \sin \pi x$$

$\therefore$

$$T_n'(t) + (n\pi\alpha)^2 T_n(t) - C_n(t) = 0$$

$$T_n(0) = \frac{2}{1} \int_0^1 \sin \pi x \cdot \sin n\pi x \, dx$$

$$\sin 2\pi x = \sum_{n=1}^{\infty} C_n \sin n\pi x$$

$$\Rightarrow \quad C_n = \frac{2}{1} \int_0^1 \sin 2\pi x \cdot \sin n\pi x \, dx$$

$$n=2 \Rightarrow \quad C_n = 2 \int_0^1 (\sin 2\pi x)^2 \, dx$$

$$T_n'(t) + (n\pi\alpha)^2 T_n = \begin{cases} 0 & \text{if } n \neq 2 \\ 1 & \text{if } n = 2 \end{cases}$$

$$T_n(0) = 2 \int_0^1 \sin \pi x \, \sin n\pi x \, dx$$

$$= \begin{cases} 0 & \text{if } n \neq 1 \\ 1 & \text{if } n = 1 \end{cases}$$

$n=1$ $$T_1'(t) + (\pi\alpha)^2 T_1 = 0$$
$$T_1(0) = 1$$

$n=2$ $$T_2' + (2\pi\alpha)^2 T_2 = 1$$
$$T_2(0) = 0$$

$n \geq 3$ $$T_n' + (n\pi\alpha)^2 T_n = 0$$
$$T_n(0) = 0$$

Solution for $n=1$

$$T_1(t) = A e^{-(\pi\alpha)^2 t} \qquad T_1(0) = 1 \Rightarrow A = 1$$

$$T_1(t) = e^{-(\pi\alpha)^2 t}$$

Solution for $n=2$

We use integration factor

$$T_2'(t) + (2\pi\alpha)^2 T_2 = 1 \quad t$$

$$\int_0^t \left(T_2(t) e^{(n\pi\alpha)^2 t} \right)' = \int_0 e^{(n\pi\alpha)^2 t}$$

$$T_2(t) e^{(n\pi\alpha)^2 t} - \cancel{T_2(0)}^{\,0} = \frac{1}{(n\pi\alpha)^2} e^{(n\pi\alpha)^2 t} \Big|_0^t$$

$$= \frac{1}{(n\pi\alpha)^2} \left(e^{(n\pi\alpha)^2 t} - 1 \right)$$

$$T_2(t) = \frac{1}{e^{(2\pi\alpha)^2 t} (2\pi\alpha)^2} \left(e^{(2\pi\alpha)^2 t} - 1 \right)$$

$$= \frac{1}{(2\pi\alpha)^2} \left(1 - e^{-(2\pi\alpha)^2 t} \right)$$

$n \geq 3 \qquad T_n(t) = A e^{-(n\pi\alpha)^2 t} \qquad T_n(0) = 0 \Rightarrow A = 0$

$$T_n(t) = 0$$

$$u(x,t) = e^{-(\pi\alpha)^2 t} \sin \pi x + \frac{1}{(2\pi\alpha)^2} \left(1 - e^{-(2\pi\alpha)^2 t} \right) \sin 2\pi x$$