MAK 501

MIDTERM 1 SOLUTION

1.

When D.E. is in the form

$$y' + p(x)y = g(x)$$

The solution is found as

$$y = e^{-\int p(x)dx}\left[C + \int g(x)e^{\int p(x)dx}dx\right] = \frac{1}{\mu(x)}\left[C + \int g(x)\mu(x)dx\right]$$

where

$$\mu(x) = e^{\int p(x)dx}$$

is the integrating factor.

$$xy' + 2y = \frac{\sin x}{x}$$

Divide by x:

$$y' + \frac{2}{x}y = \frac{\sin x}{x^2}$$

$$\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$y = \frac{1}{x^2}\left[C + \int \frac{\sin x}{x^2}x^2 dx\right] = \frac{C}{x^2} - \frac{\cos x}{x^2}$$

Initial value:

$$y(\pi) = 0 = \frac{C}{\pi^2} - \frac{\cos \pi}{\pi^2}$$

$$y = -\frac{1}{x^2} - \frac{\cos x}{x^2} = -\frac{1 + \cos x}{x^2}$$

2.

a.

$$\frac{\delta u}{\delta t} - 2\frac{\delta u}{\delta x} = 2 \tag{1}$$

Linear change of variables:

$$\alpha = ax + bt$$

$$\beta = cx + dt$$

Chain rule in two dimensions:

$$\frac{\delta u}{\delta x} = \frac{\delta u}{\delta \alpha}\frac{\delta \alpha}{\delta x} + \frac{\delta u}{\delta \beta}\frac{\delta \beta}{\delta x} = a\frac{\delta u}{\delta \alpha} + c\frac{\delta u}{\delta \beta} \tag{2}$$

$$\frac{\delta u}{\delta t} = \frac{\delta u}{\delta \alpha}\frac{\delta \alpha}{\delta t} + \frac{\delta u}{\delta \beta}\frac{\delta \beta}{\delta t} = b\frac{\delta u}{\delta \alpha} + d\frac{\delta u}{\delta \beta} \tag{3}$$

Substitute 2 and 3 into 1

$$b\frac{\delta u}{\delta \alpha} + d\frac{\delta u}{\delta \beta} - 2\left[a\frac{\delta u}{\delta \alpha} + c\frac{\delta u}{\delta \beta}\right] = 2$$

$$\frac{\delta u}{\delta \alpha}(b - 2a) + \frac{\delta u}{\delta \beta}(d - 2c) = 2$$

Let $a = 0$, $b = 2$, $c = 1$, $d = 2$ to get rid of $\frac{\delta u}{\delta \beta}$

$$2\frac{\delta u}{\delta \alpha} = 2$$

$$\frac{\delta u}{\delta \alpha} = 1$$

$$u(\alpha, \beta) = \alpha + f(\beta)$$

$$u(x, t) = 2t + f(x + 2t)$$

where f is any arbitrary function.

b.

Solution procedure is given below:

$$\frac{\delta u}{\delta x} + p(x,y)\frac{\delta u}{\delta y} = 0$$

$$\frac{dy}{dx} = p(x,y)$$

$$y = \int p(x,y)dx$$

$$C = y - \int p(x,y)dx = \Phi(x,y)$$

$$u(x,y) = f(y - \int p(x,y)dx)$$

So,

$$\frac{\delta u}{\delta x} + \frac{2}{x}\frac{\delta u}{\delta y} = 0$$

$$\frac{dy}{dx} = \frac{2}{x} \Rightarrow y = 2\ln x + C$$

$$C = y - 2\ln x = \Phi(x,y)$$

$$u(x,y) = f(y - 2\ln x)$$

where f is any arbitrary function.

3.

$$f(x) = \begin{cases} x+1 & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

a. The function is $2p$-periodic with $2p = 2$

b.

$$\begin{aligned} a_0 &= \frac{1}{2p}\int_{-p}^{p} f(x)dx \\ &= \frac{1}{2}\left[\int_0^1 (x+1)dx + \int_1^2 0dx\right] \\ &= \frac{1}{2}\left[\frac{x^2}{2} + x\right] \\ &= \frac{1}{2}\left[\frac{1}{2} + 1 - 0 - 0\right] \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi x}{p}dx \\ &= \int_0^1 (x+1)\cos n\pi x dx + \int_1^2 0\cos n\pi x dx \\ &= \int_0^1 x\cos n\pi x dx + \int_0^1 \cos n\pi x dx \\ &= \frac{x}{n\pi}\sin n\pi x\Big|_0^1 + \frac{1}{n^2\pi^2}\cos n\pi x\Big|_0^1 + \frac{1}{n\pi}\sin n\pi x\Big|_0^1 \\ &= \frac{1}{n\pi}0 - 0 + \frac{1}{n^2\pi^2}\left[\cos n\pi - 1\right] + \frac{1}{n\pi}\left[0 - 0\right] \\ &= \frac{1}{n^2\pi^2}\left[-1^n - 1\right] \end{aligned}$$

$$a_n = \begin{cases} \frac{-2}{n^2\pi^2} & n = odd \\ 0 & n = even \end{cases}$$

$$
\begin{aligned}
b_n &= \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}dx \\
&= \int_0^1 (x+1)\sin n\pi x dx + \int_1^2 0\sin n\pi x dx \\
&= \int_0^1 x\sin n\pi x dx + \int_0^1 \sin n\pi x dx \\
&= -\frac{x}{n\pi}\cos n\pi x\Big|_0^1 + \frac{1}{n^2\pi^2}\sin n\pi x\Big|_0^1 - \frac{1}{n\pi}\cos n\pi x\Big|_0^1 \\
&= -\frac{1}{n\pi} + 0 + \frac{1}{n^2\pi^2}[0-0] - \frac{1}{n\pi}[\cos n\pi - 1] \\
&= -\frac{1}{n\pi} - \frac{1}{n\pi}[-1^n - 1] \\
&= \frac{1}{n\pi}[-2(-1)^n + 1]
\end{aligned}
$$

$$
\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{p} + b_n\sin\frac{n\pi x}{p}\right] \\
&= \frac{3}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi^2}(-1^n - 1)\cos n\pi x + \frac{1}{n\pi}(-2(-1)^n + 1)\sin n\pi x\right]
\end{aligned}
$$

$x = 1$:

$$
\begin{aligned}
f(x) = &= \frac{3}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi^2}(-1^n - 1)\cos n\pi x + \frac{1}{n\pi}(-2(-1)^n + 1)\sin n\pi x\right] \\
&= \frac{3}{4} + \sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}(-1^n - 1)\cos n\pi \\
&= \frac{3}{4} - \frac{2}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos(2k+1)\pi \\
&= \frac{3}{4} + \frac{2}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}
\end{aligned}
$$

$n = 1 \rightarrow k = 0$

$$
\begin{aligned}
f(x) &= \frac{3}{4} + \frac{2}{\pi^2} \\
&= 0.9526
\end{aligned}
$$

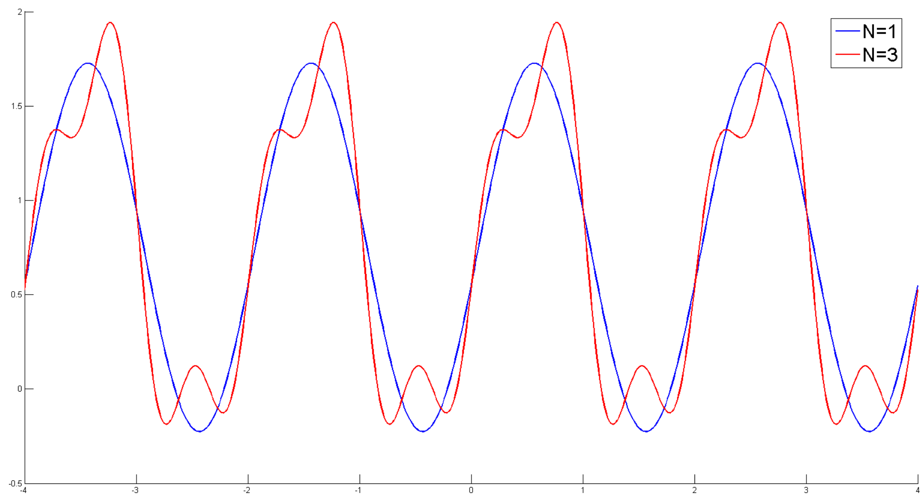
$n = 3 \rightarrow k = 1$

$$f(x) = \frac{3}{4} + \frac{2}{\pi^2} + \frac{2}{\pi^2}\frac{1}{3^2}$$
$$= 0.9752$$

$n = 5 \rightarrow k = 2$

$$f(x) = \frac{3}{4} + \frac{2}{\pi^2} + \frac{2}{\pi^2}\frac{1}{3^2} + \frac{2}{\pi^2}\frac{1}{5^2}$$
$$= 0.9833$$

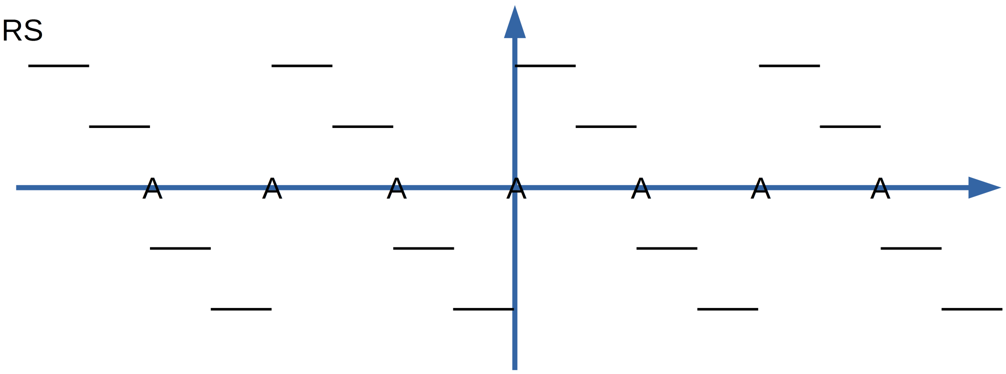
c.



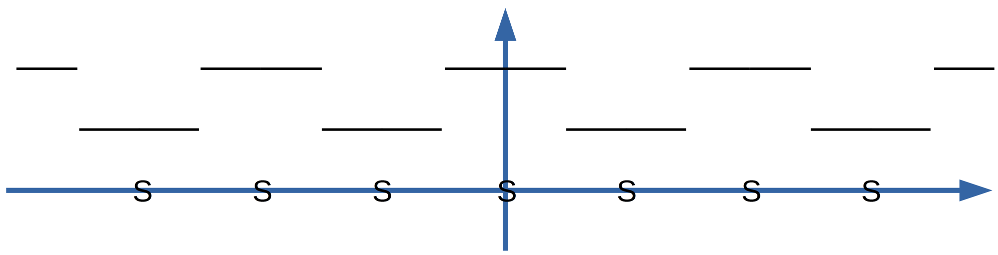
4.

$$f(x) = \begin{cases} 2 & 0 < x < 2 \\ 1 & 2 < x < 4 \end{cases}$$

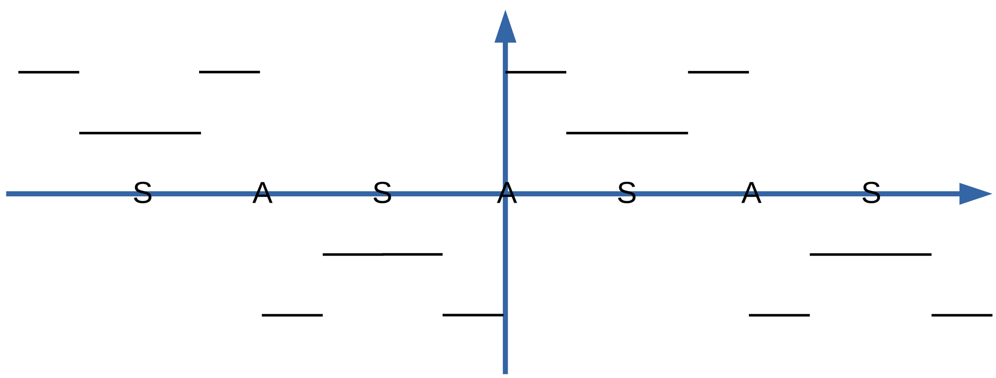
HRS



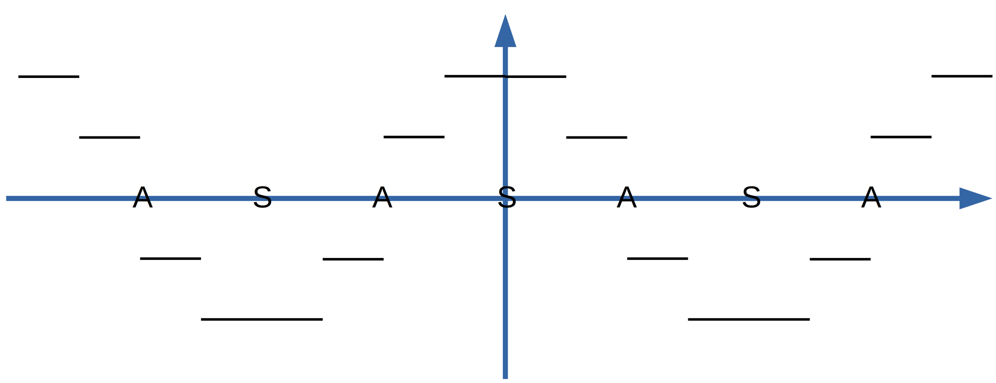
HRC



QRS



QRC



$$L = 4$$

<u>HRS</u>

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad 0 < x < L$$

$$\begin{aligned}
b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \\
&= \frac{2}{4} \left[\int_0^2 2 \sin \frac{n\pi x}{4} dx + \int_2^4 \sin \frac{n\pi x}{4} dx \right] \\
&= \frac{1}{2} \left[-\frac{8}{n\pi} \cos \frac{n\pi x}{4} \bigg|_0^2 - \frac{4}{n\pi} \cos \frac{n\pi x}{4} \bigg|_2^4 \right] \\
&= \frac{1}{2} \left[-\frac{8}{n\pi} \left(\cos \frac{n\pi}{2} - 1 \right) - \frac{4}{n\pi} \left(\cos n\pi - \cos \frac{n\pi}{2} \right) \right] \\
&= \frac{4}{n\pi} - \frac{2}{n\pi} \cos \frac{n\pi}{2} - \frac{2}{n\pi} \cos n\pi \\
&= \frac{4}{n\pi} - \frac{2}{n\pi} \cos \frac{n\pi}{2} - \frac{2}{n\pi} (-1^n)
\end{aligned}$$

$$f(x) = \sum_{n=1}^{\infty} \left[\frac{4}{n\pi} - \frac{2}{n\pi} \cos \frac{n\pi}{2} - \frac{2}{n\pi} (-1^n) \right] \sin \frac{n\pi x}{4}$$

HRC

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \quad 0 < x < L$$

$$\begin{aligned} a_0 &= \frac{1}{L}\int_0^L f(x)dx \\ &= \frac{1}{4}\left[\int_0^2 2dx + \int_2^4 1dx\right] \\ &= \frac{1}{4}\left[2x\Big|_0^2 + x\Big|_2^4\right] \\ &= \frac{1}{4}\left[4 - 0 + 4 - 2\right] \\ &= \frac{3}{2} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \\ &= \frac{2}{4}\left[\int_0^2 2\cos\frac{n\pi x}{4}dx + \int_2^4 \cos\frac{n\pi x}{4}dx\right] \\ &= \frac{1}{2}\left[\frac{8}{n\pi}\sin\frac{n\pi x}{4}\Big|_0^2 + \frac{4}{n\pi}\sin\frac{n\pi x}{4}\Big|_2^4\right] \\ &= \frac{1}{2}\left[\frac{8}{n\pi}\left(\sin\frac{n\pi}{2} - 0\right) + \frac{4}{n\pi}\left(0 - \sin\frac{n\pi}{2}\right)\right] \\ &= \frac{2}{n\pi}\sin\frac{n\pi}{2} \end{aligned}$$

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{n\pi}\sin\frac{n\pi}{2}\right]\cos\frac{n\pi x}{4}$$

QRS

$$f(x) = \sum_{n=1,3}^{\infty} b_n \sin \frac{n\pi x}{2L}, \quad 0 < x < L$$

$$\begin{aligned}
b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{2L} dx \\
&= \frac{2}{4} \left[\int_0^2 2 \sin \frac{n\pi x}{8} dx + \int_2^4 \sin \frac{n\pi x}{8} dx \right] \\
&= \frac{1}{2} \left[-\frac{16}{n\pi} \cos \frac{n\pi x}{8} \bigg|_0^2 - \frac{8}{n\pi} \cos \frac{n\pi x}{8} \bigg|_2^4 \right] \\
&= \frac{1}{2} \left[-\frac{16}{n\pi} \left(\cos \frac{n\pi}{4} - 1 \right) - \frac{8}{n\pi} \left(\cos \frac{n\pi}{2} - \cos \frac{n\pi}{4} \right) \right] \\
&= \frac{8}{n\pi} - \frac{4}{n\pi} \cos \frac{n\pi}{4} - \frac{4}{n\pi} \cos \frac{n\pi}{2}
\end{aligned}$$

$$\begin{aligned}
f(x) &= \sum_{n=1,3}^{\infty} \left[\frac{8}{n\pi} - \frac{4}{n\pi} \cos \frac{n\pi}{4} \underbrace{- \frac{4}{n\pi} \cos \frac{n\pi}{2}}_{\text{=0 for n=1,3,...}} \right] \sin \frac{n\pi x}{8} \\
&= \sum_{n=1,3}^{\infty} \left[\frac{8}{n\pi} - \frac{4}{n\pi} \cos \frac{n\pi}{4} \right] \sin \frac{n\pi x}{8}
\end{aligned}$$

<u>QRC</u>

$$f(x) = \sum_{n=1,3}^{\infty} a_n \cos \frac{n\pi x}{2L}, \quad 0 < x < L$$

$$\begin{aligned}
b_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{2L} dx \\
&= \frac{2}{4} \left[\int_0^2 2 \cos \frac{n\pi x}{8} dx + \int_2^4 \cos \frac{n\pi x}{8} dx \right] \\
&= \frac{1}{2} \left[\frac{16}{n\pi} \sin \frac{n\pi x}{8} \bigg|_0^2 + \frac{8}{n\pi} \sin \frac{n\pi x}{8} \bigg|_2^4 \right] \\
&= \frac{1}{2} \left[\frac{16}{n\pi} \left(\sin \frac{n\pi}{4} - 0 \right) + \frac{8}{n\pi} \left(\sin \frac{n\pi}{2} - \sin \frac{n\pi}{4} \right) \right] \\
&= \frac{4}{n\pi} \sin \frac{n\pi}{4} + \frac{4}{n\pi} \sin \frac{n\pi}{2} \\
&= \frac{4}{n\pi} \sin \frac{n\pi}{4} + \frac{4}{n\pi} (-1^{n+1})
\end{aligned}$$

$$f(x) = \sum_{n=1,3}^{\infty} \left[\frac{4}{n\pi} \sin \frac{n\pi}{4} + \frac{4}{n\pi} (-1^{n+1}) \right] \cos \frac{n\pi x}{8}$$