TOBB EKONOMİ VE TEKNOLOJİ ÜNİVERSİTESİ

MAK 501 ENGINEERING MATHEMATICS

FALL 2016

Due Date: 12.12.2016- Monday* (18:30)

HOMEWORK 6

1.

a In the formula $f(x) = \frac{1}{2\pi}\int \hat{f}(\omega)e^{i\omega x}d\omega$, set $x \to -\omega$ and $\omega \to x$, obtain

$$f(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(x)e^{-i\omega x}dx = \frac{1}{2\pi}F\{f(\hat{x})\}$$

or

$$F\{\hat{f}(x)\} = \frac{1}{2\pi}f(-\omega)$$

Equivalently,

$$F^{-1}\{f(-\omega)\} = \frac{1}{2\pi}\hat{f}(x)$$

b Entry 4 in Appendix D (Greenberg): $F\{\underbrace{e^{-a|x|}}_{f(x)}\} = \underbrace{\frac{2a}{\omega^2 + a^2}}_{\hat{f}(w)}$ So, $\hat{f}(x) = \frac{2a}{x^2+a^2}$ and $f(-\omega) = e^{-a|-\omega|} = e^{-a|\omega|}$. Thus, $F\{\hat{f}(x)\} = 2\pi f(-\omega)$ gives

$$F\left\{\frac{2a}{x^2 + a^2}\right\} = 2\pi e^{-a|\omega|}$$

c Entry 9 in Appendix D (Greenberg): $F\{\underbrace{H(x+a) - H(x-a)}_{f(x)}\} = \underbrace{\frac{2\sin\omega a}{\omega}}_{f(\omega)}$. So, $\hat{f}(x) = \frac{2\sin xa}{x}$ and $f(-\omega) = H(-\omega + a) - H(-\omega - a)$. Thus, 1 gives

$$F\left\{\frac{2\sin xa}{x}\right\} = 2\pi\left[H(-\omega + a) - H(-\omega - a)\right]$$

or, since $H(-x) = 1 - H(x)$ and by linearity to cancel the 2's,

$$F\left\{\frac{\sin xa}{x}\right\} = \pi\left[1 - H(\omega - a) - 1 + H(\omega + a)\right]$$

$$= \pi\left[H(\omega + a) - H(\omega - a)\right]$$

d Entry 3 in Appendix D (Greenberg): $F\{\underbrace{H(-x)e^{ax}}_{f(x)}\} = \underbrace{\frac{1}{a - i\omega}}_{\hat{f}(\omega)}$. So, $\hat{f}(x) = \frac{1}{a-ix}$ and $f(-\omega) = H(-\omega)e^{-a\omega}$. Thus, 1 gives

$$F\left\{\frac{1}{a - ix}\right\} = 2\pi H(\omega)e^{-a\omega}$$

2.

$$\alpha^2 u_{xx} = u_t + V u_x, \quad (-\infty < x < \infty, 0 < t < \infty)$$

$$u(x,0) = f(x), \quad (-\infty < x < \infty)$$

Fourier transform:

$$\begin{aligned}
\alpha^2 (i\omega^2)\hat{u} &= \hat{u}_t + i\omega V \hat{u} \\
\hat{u}_t + (\alpha^2\omega^2 + i\omega V)\hat{u} &= 0 \\
\hat{u}(\omega, t) &= A e^{-(\alpha^2\omega^2 + i\omega V)t} \\
\hat{u}(\omega, 0) &= \hat{f}(\omega) = A \\
\hat{u}(\omega, t) &= \hat{f}(\omega) e^{-(\alpha^2\omega^2 + i\omega V)t}
\end{aligned}$$

From Appendix D (Greenberg):

Entry 6:	$e^{-(\alpha^2\omega^2)t}$	$\frac{1}{2(\alpha\sqrt{t})\sqrt{\pi}} e^{-\frac{x^2}{4\alpha^2}t}$
Entry 11 ($a = 1, b = -Vt$)	$e^{-(\alpha^2\omega^2)t} e^{-iVt\omega}$	$\frac{1}{2(\alpha\sqrt{t})\sqrt{\pi}} e^{-(x-Vt)^2/4\alpha^2 t}$

$$\begin{aligned}
u(x,t) &= f(x) * \frac{1}{2(\alpha\sqrt{t})\sqrt{\pi}} e^{-(x-Vt)^2/4\alpha^2 t} \\
&= \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} f(\xi) e^{-(x-\xi-Vt)^2/4\alpha^2 t} d\xi
\end{aligned}$$

3.

$$u'' - 9u = 50e^{-2x}$$

$$s^2\bar{u} - su(0) - u'(0) - 9\bar{u} = \frac{50}{s+2}$$

$$(s^2 - 9)\bar{u} = u_0 s + u'(0) + \frac{50}{s+2}$$

$$\bar{u} = u_0\frac{s}{s^2-9} + \frac{u'(0)}{s^2-9} + \frac{50}{(s^2-9)(s+2)}$$

$$u(x) = u_0 \cosh 3x + u'(0)\frac{\sinh 3x}{3} + 50\frac{\sinh 3x}{3}e^{-2x}$$

$$\begin{aligned}\sinh 3x * e^{-2x} &= \int_0^x \sinh 3\xi e^{-2(x-\xi)} d\xi \\ &= e^{-2x}\int_0^x \frac{e^{5\xi} - e^{-\xi}}{2} d\xi \\ &= \frac{e^{3x}}{10} + \frac{e^{-3x}}{2} - \frac{3}{5}e^{-2x}\end{aligned}$$

$$u(x) = u_0\frac{e^{3x}+e^{-3x}}{2} + u'(0)\frac{e^{3x}-e^{-3x}}{6} + \frac{50}{3}\left[\frac{e^{3x}}{10} + \frac{e^{-3x}}{2} - \frac{3}{5}e^{-2x}\right]$$

To satisfy the condition $u(\infty)$ bounded, set the coefficient of e^{3x} to 0, and $u(\infty)$ becomes:

$$0 = \frac{u_0}{2} + \frac{u'}{6} + \frac{5}{3}$$

$$u' = -3u_0 - 10$$

So,

$$\begin{aligned}u(x) &= \left[\frac{u_0}{2} - \frac{-3u_0 - 10}{6} + \frac{25}{3}\right] e^{-3x} - 10e^{-2x} \\ &= (u_0 + 10)e^{-3x} - 10e^{-2x}\end{aligned}$$

4. The right hand side can be expressed in terms of a step function

$$f(t) = e^{-t}H(t-3)$$

Its Laplace transform is

$$F(s) = G(s)e^{-3s}$$

where $g(t-3) = e^{-t}$ or $g(t) = e^{-(t+3)}$, so that $G(s) = \frac{e^{-3}}{s+1}$. Thus,

$$F(s) = \frac{e^{-3(s+1)}}{s+1}$$

By taking the Laplace transform of the differential equation, obtain

$$s^2Y(s) - s3 - 7 + 5\,[sY(s) - 3] - 6Y(s) = \frac{e^{-3(s+1)}}{s+1}$$

$$Y(s) = \frac{3s+22}{s^2+5s-6} + \frac{e^{-3(s+1)}}{(s+1)(s^2+5s-6)}$$

Invert these separately:

$$Y_1(s) = \frac{3s+22}{s^2+5s-6} = \frac{3s+22}{(s+6)(s-1)} = \frac{a}{s+6} + \frac{b}{s-1}$$

$$b = \frac{25}{7}, \quad a = -\frac{4}{7}$$

$$Y_1(s) = \frac{-\frac{4}{7}}{s+6} + \frac{\frac{25}{7}}{s-1}$$

$$y_1(t) = \frac{25}{7}e^{t} - \frac{4}{7}e^{-6t}$$

For the other term, we first need:

$$G_2(s) = \frac{1}{(s+1)(s^2+5s-6)} = \frac{1}{(s+1)(s+6)(s-1)} = \left(\frac{c}{s+1} + \frac{d}{s+6} + \frac{e}{s-1}\right)$$

$$e = \frac{1}{14}, \quad c = -\frac{1}{10}, \quad d = \frac{1}{35}$$

$$G_2(s) = \left(\frac{-\frac{1}{10}}{s+1} + \frac{\frac{1}{35}}{s+6} + \frac{\frac{1}{14}}{s-1}\right)$$

$$g_2(t) = -\frac{1}{10}e^{-t} + \frac{1}{35}e^{-6t} + \frac{1}{14}e^{t}$$

We use the formula

$$L^{-1}[e^{-3s}G_2(s)] = H(t-3)g_2(t-3)$$

Thus

$$y(t) = \frac{25}{7}e^t - \frac{4}{7}e^{-6t} + e^{-3}H(t-3)\left[-\frac{1}{10}e - (t-3) + \frac{1}{35}e^{-6(t-3)} + \frac{1}{14}e^{t-3}\right]$$

Note that for $t > 3$

$$y(t) = e^t\left[\frac{25}{7} + \frac{e^{-6}}{14}\right] + e^{-6t}\left[\frac{e^{15}}{35} - \frac{4}{7}\right] + \frac{1}{10}e^{-t}$$