# TOBB EKONOMİ VE TEKNOLOJİ ÜNİVERSİTESİ

# MAK 501 ENGINEERING MATHEMATICS

# FALL 2016

# HOMEWORK 5 SOLUTIONS

**1.**

$h(z) = \sinh\sqrt{\lambda}(z-H)$. The condition at $z=0$ implies $m=7$ only (and $\sin 7\theta$ only also). Thus by superposition

$$u(r,\theta,z) = \sum_{n=1}^{\infty} A_{mn} \sinh\sqrt{\lambda_{7n}}(H-z)\sin 7\theta J_7\left(\sqrt{\lambda_{7n}}r\right),$$

The boundary condition at $z=0$ (after cancelling $\sin 7\theta$) determines $A_n$

$$A_n \sinh\sqrt{\lambda_{7n}}H = \frac{\int_0^a \alpha(r) J_7\left(\sqrt{\lambda_{7n}}r\right)\ r\ dr}{\int_0^a J_7^2\left(\sqrt{\lambda_{7n}}r\right)\ r\ dr}.$$

**2.**

By separation of variables, time only first, $\nabla_3^2\phi + \lambda\phi = 0$ and $dh/dt = -\lambda kh$.

If we next separate $z$, we know $Q_{zz} = -\nu Q$

with $\nu = (\ell\pi/H)^2$, because of the boundary conditions and $Q(z) = \cos\ell\pi z/H, \ell = 0,1,2,\ldots$. Thus $\nabla^2\phi + \lambda'\phi = 0$ where $\lambda' = \lambda - (\ell\pi/H)^2$ and here the Laplacian is the two-dimensional one

The boundary condition is $g'(0) = g'(\pi/2)$

This is a cosine series in $\theta$, where $\mu = [m\pi/(\pi/2)]^2 = 4m^2$ and $g(\theta) = \cos 2m\theta, m = 0,1,2,\ldots$

Note that $\lambda' = 0$ only for $m=0$ with $f(r) = 1$.

The boundary condition at $r=a$ yields $J'_{2m}\left(\sqrt{\lambda'_{mn}}a\right) = 0$.

three-dimensional eigenfunctions are

$$\phi_{\ell mn}(r,\theta,z) = \begin{cases} 1 & m=0, \ell=0, n=1 \\ \cos\frac{\ell\pi z}{H}\cos 2m\theta J_{2m}\left(\sqrt{\lambda'_{mn}}r\right) & \text{otherwise.} \end{cases}$$

By superposition

$$u(r,\theta,z,t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty} A_{\ell mn}\phi_{\ell mn}(r,\theta,z)e^{-\lambda_{\ell mn}kt},$$

where $\lambda_{\ell mn} = (\ell\pi/H)^2 + \lambda'_{mn}$. The initial condition determines $A_{\ell mn}$

$$A_{\ell mn} = \int\int\int f(r,\theta,z)\phi_{\ell mn}(r,\theta,z)\ rdr\ d\theta\ dz \Big/ \int\int\int \phi^2_{\ell mn}\ rdr\ d\theta\ dz\ .$$

As $t \to \infty, e^{-\lambda_{\ell mn}kt} \to \infty$ except for $\ell = 0, m = 0, n = 1$ since $\lambda_{001} = 0$ :

$$u(r,\theta,z,t) \to A_{001} = \frac{1}{H\pi a^2/4}\int\int\int f(r,\theta,z)r\ dr\ d\theta\ dz,$$

the average of the initial temperature distribution. We expect this because with insulated boundaries the equilibrium temperature ought to be a constant (with the same thermal energy as the initial condition).

**3.**

Try u as,

$$u(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn}r)(A_{mn}\cos m\theta + B_{mn}\sin m\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r,\theta),$$

where $\phi_{mn}(r,\theta) = J_m(\lambda_{mn}r)(A_{mn}\cos m\theta + B_{mn}\sin m\theta)$. We plug this solution into the equation, use the fact that $\nabla^2(\phi_{mn}) = -\lambda^2_{mn}\phi_{mn} = -\alpha^2_{mn}\phi_{mn}$, and get

$$\begin{aligned}\nabla^2\left(\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r,\theta)\right) &= 2 + r^3\cos 3\theta \\ &\Rightarrow \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\nabla^2\left(\phi_{mn}(r,\theta)\right) = 2 + r^3\cos 3\theta \\ &\Rightarrow \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} -\alpha^2_{mn}\phi_{mn}(r,\theta) = 2 + r^3\cos 3\theta.\end{aligned}$$

We recognize this expansion as the expansion of the function $2 + r^3\cos 3\theta$ in terms of the functions $\phi_{mn}$. Because of the special form of the right side, it follows that all $A_{mn}$ and $B_{mn}$ are zero, except $A_{0,n}$ and $A_{3,n}$. So

$$\sum_{n=1}^{\infty} -\alpha^2_{0n}A_{0,n}J_0(\alpha_{0n}r) = 2$$

and

$$\cos 3\theta\sum_{n=1}^{\infty} -\alpha^2_{3n}A_{3,n}J_3(\alpha_{3n}r) = r^3\cos 3\theta.$$

Using the series

$$1 = \sum_{n=1}^{\infty}\frac{2}{\alpha_{0,n}J_1(\alpha_{0,n})}J_0(\alpha_{0,n}r) \quad 0 < r < 1,$$

we find that

$$2 = \sum_{n=1}^{\infty}\frac{4}{\alpha_{0,n}J_1(\alpha_{0,n})}J_0(\alpha_{0,n}r) \quad 0 < r < 1,$$

and so

$$\sum_{n=1}^{\infty} -\alpha^2_{0n}A_{0,n}J_0(\alpha_{0n}r) = \sum_{n=1}^{\infty}\frac{4}{\alpha_{0,n}J_1(\alpha_{0,n})}J_0(\alpha_{0,n}r),$$

from which we conclude that

$$-\alpha^2_{0n}A_{0,n} = \frac{4}{\alpha_{0,n}J_1(\alpha_{0,n})}$$

or

$$A_{0,n} = \frac{-4}{\alpha_{0,n}^3 J_1(\alpha_{0,n})}.$$

From the series

$$r^3 = \sum_{n=1}^{\infty} \frac{2}{\alpha_{3,n} J_4(\alpha_{3,n})} J_3(\alpha_{3,n} r) \quad 0 < r < 1,$$

we conclude that

$$\sum_{n=1}^{\infty} \frac{2}{\alpha_{3,n} J_4(\alpha_{3,n})} J_3(\alpha_{3,n} r) = \sum_{n=1}^{\infty} -\alpha_{3n}^2 A_{3,n} J_3(\alpha_{3n} r)$$

and so

$$A_{3,n} = \frac{-2}{\alpha_{3,n}^3 J_4(\alpha_{3,n})}.$$

Hence

$$u(r,\,\theta) = \sum_{n=1}^{\infty} \frac{-4}{\alpha_{0,n}^3 J_1(\alpha_{0,n})} J_0(\alpha_{0,n} r) + \cos 3\theta \sum_{n=1}^{\infty} \frac{-2}{\alpha_{3,n}^3 J_4(\alpha_{3,n})} J_3(\alpha_{3,n} r).$$

$0<r<a \ , \ 0<\theta<2\pi$
$t>0$

4) a) $$\frac{\partial u}{\partial t}=c^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)+q(r,\theta,t) \Rightarrow u(a,\theta,t)=0 = \text{B.C.1}$$

Problemin çözümü için eigenfunction expension kullanılır.

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$u(r,\theta,0)=f(r,\theta)$ given in (I). ⇒ Lets start by assuming that u has an expension in terms of the eigenfunctions of the Helmholtz problem.

Burada $u(r,\theta)$ in spherical coordinates'da çözümleme yaparsak $r^2R''+rR'+(kr^2-m^2)R=0$ eşitliğinden → $\cos m\theta J_m(\lambda_{mn}r)$ ve $\sin m\theta J_m(\lambda_{mn}r)$ eigen function larına ulaşılır.

Bu nedenle (I), (II) ve (III)'de verilen eşitlikler kullanılarak çözümleme yapılır.

(I) $$f(r,\theta)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn}r).(a_{mn}.\cos m\theta+b_{mn}\sin m\theta)$$

⇒ Asmar Section 4.3 equation 12-14'de $a_{mn}$ ve $b_{mn}$ tanımı şöyledir;

$$a_{mn}=\frac{2}{\pi a^2 J^2_{m+1}(\alpha_{mn})}\cdot\int_0^a\int_0^{2\pi} f(r,\theta).\cos m\theta . J_m(\lambda_{mn}r)r.d\theta.dr$$ ✓

$$b_{mn}=\frac{2}{\pi.a^2.J^2_{m+1}(\alpha_{mn})}\int_0^a\int_0^{2\pi} f(r,\theta).\sin m\theta . J_m(\lambda_{mn}r).r.d\theta dr$$ ✓

(II) $$u(r,\theta,t)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn}r).(A_{mn}(t)\cos m\theta+B_{mn}(t)\sin m\theta) = \underline{\phi_{mn}}$$

$$\nabla^2 u=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} -\lambda^2_{mn}.J_m(\lambda_{mn}r).(A_{mn}(t)\cos m\theta+B_{mn}(t)\sin m\theta) = \underline{-\lambda^2_{mn}.\phi_{mn}}$$ ✓

$$\frac{\partial u}{\partial t}=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn}r).(A'_{mn}(t)\cos m\theta+B'_{mn}(t)\sin m\theta)$$ ✓

(III) $$q(r,\theta,t)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn}r)(c_{mn}(t)\cos m\theta+d_{mn}(t)\sin m\theta)$$

Türevlerden elde ettiğimiz değerleri eşitlikte yerine yazarsak,

4) b)

$$\Rightarrow\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn}r).(A'_{mn}(t).\cos m\theta+B'_{mn}(t)\sin m\theta)=c^2\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} -\lambda^2_{mn}.J_m(\lambda_{mn}r)(A_{mn}(t)\cos m\theta+B_{mn}(t)\sin m\theta)+$$
$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn}r)\,c_{mn}(t)\cos m\theta+d_{mn}(t).\sin m\theta)$$

$$\Rightarrow A'_{mn}(t)+c^2\lambda^2_{mn}A_{mn}(t)=c_{mn}(t)$$
$$B'_{mn}(t)+c^2\lambda^2_{mn}B_{mn}(t)=d_{mn}(t)$$

for $c=1$ soruda verilen eşitlikler elde edilebilir.

$$u(r,\theta,0)=f(r,\theta) \Rightarrow \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn}r)(a_{mn}\cos m\theta+b_{mn}\sin m\theta)$$
$$=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn}r).(A_{mn}(0)\cos m\theta+B_{mn}(0)\sin m\theta)$$

$$\Rightarrow A_{mn}(0)=a_{mn} \qquad B_{mn}(0)=b_{mn}$$

4°) c) $$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) + q(r,\theta,t)$$

$u(a,\theta,t)=0$
$u(r,\theta,0)=f(r,\theta)$
$0<r<a$
$0<\theta<2\pi$
$t>0$

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)\left(A_{mn}(t)\cos m\theta + B_{mn}(t)\sin m\theta\right)$$

where $\lambda_{mn} = \dfrac{a_{m+1/2J}}{a}$

b şıkkında elde ettiğimiz equation da homojen kısmı için

$A_{mn_h}(t) = xe^{\alpha t}$ dersek $\Rightarrow 0 = x\cdot\alpha e^{\alpha t} + \lambda_{mn}^2 c^2 x e^{\alpha t}$

for $\alpha = -\lambda_{mn}^2 c^2$

$\Rightarrow A_{mn_h}(t) = x\cdot e^{-\lambda^2_{mn} c^2 t}$

IC. $\Rightarrow$ for $t=0$ $x = a_{mn}$, $c=1$ alırsak $\Rightarrow$ $A_{mn_h}(t) = a_{mn}\cdot e^{-\lambda_{mn}^2 t}$

$$A_{mn_p}(t) = e^{-\lambda_{mn}^2 t}\int_0^t e^{+\lambda_{mn}^2 s}\cdot c_{mn}(s)\cdot ds$$

$$\Rightarrow c_{mn}(t) = -\lambda_{mn}^2\cdot e^{-\lambda_{mn}^2 t}\int_0^t e^{-\lambda_{mn}^2 s} c_{mn}(s)\cdot ds + e^{-\lambda_{mn}^2 t}\cdot\left\{e^{+\lambda_{mn}^2 t}\cdot c_{mn}(t) - c_{mn}(0)\right\}$$

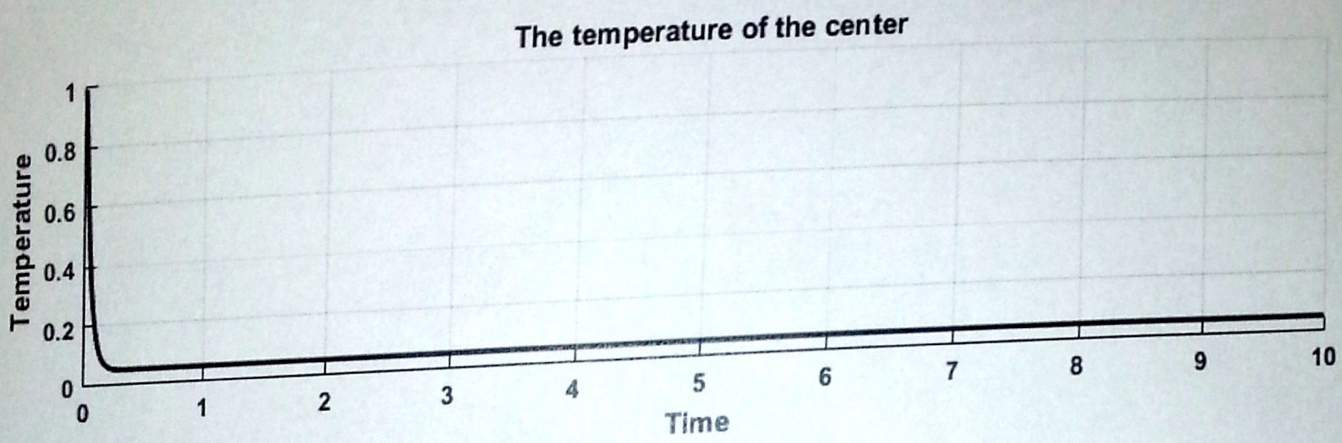
$$+ \lambda_{mn}^2 e^{-\lambda^2_{mn} t}\int_0^t e^{\lambda^2_{mn} s} c_{mn}(s)\cdot ds$$

$\Rightarrow c_{mn}(t) = c_{mn}(t)$ eşitliği sağlanır

$$\Rightarrow A_{mn}(t) = A_{mn_h}(t) + A_{mn_p}(t) = e^{-\lambda_{mn}^2 t}\cdot\left\{a_{mn} + \int_0^t e^{\lambda^2_{mn} s} c_{mn}(s)\cdot ds\right.$$

Benzer eşitlik $B_{mn}$ için yazılır ve $d_{mn}(t)$ gözölürse,

$$B_{mn}(t) = B_{mn_h}(t) + B_{mn_p}(t) = e^{-\lambda^2_{mn} t}\cdot\left\{b_{mn} + \int_0^t e^{\lambda^2_{mn} s}\cdot d_{mn}(s)\cdot ds\right.$$




$$t \to \infty \quad \Rightarrow \quad u \approx 0.0469$$