TOBB EKONOMİ VE TEKNOLOJİ ÜNİVERSİTESİ

MAK 501 ENGINEERING MATHEMATICS

FALL 2016

HOMEWORK 4 SOLUTIONS

1.

a) Consider the partial differential equation, $\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}$.

The damping term $-\beta \frac{\partial u}{\partial t}$ should be added to the acceleration $\frac{\partial^2 u}{\partial t^2}$, if the velocity $\frac{\partial u}{\partial t}$ is negative and should be subtracted from the acceleration if the velocity is positive.

It thereby decreases the speed to "dampen" the motion.

This is achieved if and only if $\beta > 0$.

(b) By separation of variables, $u = \phi(x)h(t)$, $\frac{\rho_0 h'' + \beta h'}{h T_0} = \frac{\phi''}{\phi} = -\lambda$. The boundary conditions $\phi(0) = 0$ and $\phi(L) = 0$ yield $\lambda = (n\pi/L)^2$ with $\phi = \sin n\pi x/L$, $n = 1, 2, 3, \ldots$. The time-dependent equation has constant coefficients,

$$\rho_0 h'' + \beta h' + \left(\frac{n\pi}{L}\right)^2 T_0 h = 0,$$

and hence can be solved by substitution $h = e^{rt}$. This yields the quadratic equation

$$\rho_0 r^2 + \beta r + \left(\frac{n\pi}{L}\right)^2 T_0 = 0,$$

whose roots are

$$r = \frac{-\beta \pm \sqrt{\beta^2 - 4\rho_0 T_0 (n\pi/L)^2}}{2\rho_0}.$$

Since $\beta^2 < 4\rho_0 T_0 (\pi/L)^2$, the discriminant is < 0 for all n:

$$r = -\frac{\beta}{2\rho_0} + i w_n, \text{ where } w_n = \sqrt{\frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2 - \frac{\beta^2}{4\rho_0^2}}.$$

Real solutions are $h = e^{-\beta t/2\rho_0}(\sin w_n t, \cos w_n t)$. Thus by superposition

$$u = e^{-\beta t/2\rho_0} \sum_{n=1}^{\infty} (a_n \cos w_n t + b_n \sin w_n t) \sin \frac{n\pi x}{L}.$$

The initial condition $u(x,0) = f(x)$ determines a_n, $a_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} dx$, while $\frac{\partial u}{\partial t}(x,0) = g(x)$ is a little more complicated, $g(x) = \sum_{n=1}^{\infty} b_n w_n \sin \frac{n\pi x}{L} - \frac{\beta}{2\rho_0} \underbrace{\sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}}_{f(x)}$, and thus

$$b_n w_n = \frac{\beta a_n}{2\rho_0} + \frac{2}{L}\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx.$$

2.

(e) Separation of variables, $u = \phi(x)h(y)$, yields the eigenvalues $\lambda = (n\pi/L)^2$ and corresponding eigenfunctions $\phi = \sin n\pi x/L, n = 1, 2, 3, \ldots$ The y-dependent differential equation, $\frac{d^2h}{dy^2} = \left(\frac{n\pi}{L}\right)^2 h$, satisfies $h(0) - \frac{dh}{dy}(0) = 0$. The general solution $h = c_1 \cosh \frac{n\pi y}{L} + c_2 \sinh \frac{n\pi y}{L}$ obeys $h(0) = c_1$, while $\frac{dh}{dy} = \frac{n\pi}{L}\left(c_1 \sinh \frac{n\pi y}{L} + c_2 \cosh \frac{n\pi y}{L}\right)$ obeys $\frac{dh}{dy}(0) = c_2 \frac{n\pi}{L}$. Thus, $c_1 = c_2 \frac{n\pi}{L}$ and hence $h_n(y) = \cosh \frac{n\pi y}{L} + \frac{L}{n\pi} \sinh \frac{n\pi y}{L}$. Superposition yields

$$u(x, y) = \sum_{n=1}^{\infty} A_n h_n(y) \sin n\pi x/L,$$

where A_n is determined from the boundary condition, $f(x) = \sum_{n=1}^{\infty} A_n h_n(H) \sin n\pi x/L$, and hence

$$A_n h_n(H) = \frac{2}{L} \int_0^L f(x) \sin n\pi x/L \, dx \, .$$

3. a)

$$\frac{d^2\emptyset}{d\theta^2} = -\lambda\emptyset$$
$$\emptyset(-\pi) = \emptyset(\pi)$$
$$\frac{d\emptyset}{d\theta}(-\pi) = \frac{d\emptyset}{d\theta}(-\pi)$$

Boundary conditions are replaced by,
$\emptyset(0) = 0$ and $\emptyset(\pi/2) = 0$. This makes,

sine series with $L = \pi/2$ so that $\lambda = (n\pi/L)^2 = (2n)^2$ and the eigenfunctions are $\phi = \sin n\pi\theta/L = \sin 2n\theta, n = 1, 2, 3, \ldots$. The radial part which is zero at $r = a$ is $G = (r/a)^{2n} - (a/r)^{2n}$. Thus by superposition,

$$u(r, \theta) = \sum_{n=1}^{\infty} A_n \left[\left(\frac{r}{a}\right)^{2n} - \left(\frac{a}{r}\right)^{2n}\right] \sin 2n\theta.$$

The nonhomogeneous boundary condition, $f(\theta) = \sum_{n=1}^{\infty} A_n \left[\left(\frac{b}{a}\right)^{2n} - \left(\frac{a}{b}\right)^{2n}\right] \sin 2n\theta$, determines A_n : $A_n \left[\left(\frac{b}{a}\right)^{2n} - \left(\frac{a}{b}\right)^{2n}\right] = \frac{4}{\pi} \int_0^{\pi/2} f(\theta) \sin 2n\theta \, d\theta$.

(b) The two homogeneous boundary conditions are in r, and hence $\phi(r)$ must be an eigenvalue problem. By separation of variables, $u = \phi(r)G(\theta)$, $d^2G/d\theta^2 = \lambda G$ and $r^2\frac{d^2\phi}{dr^2} + r\frac{d\phi}{dr} + \lambda\phi = 0$. The radial equation is equidimensional (see p.78) and solutions are in the form $\phi = r^p$. Thus $p^2 = -\lambda$ (with $\lambda > 0$) so that $p = \pm i\sqrt{\lambda}$. $r^{\pm i\sqrt{\lambda}} = e^{\pm i\sqrt{\lambda}\ln r}$. Thus real solutions are $\cos(\sqrt{\lambda}\ln r)$ and $\sin(\sqrt{\lambda}\ln r)$. It is more convenient to use independent solutions which simplify at $r = a$, $\cos[\sqrt{\lambda}\ln(r/a)]$ and $\sin[\sqrt{\lambda}\ln(r/a)]$. Thus the general solution is

$$\phi = c_1\cos[\sqrt{\lambda}\ln(r/a)] + c_2\sin[\sqrt{\lambda}\ln(r/a)].$$

The homogeneous condition $\phi(a) = 0$ yields $0 = c_1$, while $\phi(b) = 0$ implies $\sin[\sqrt{\lambda}\ln(r/a)] = 0$. Thus $\sqrt{\lambda}\ln(b/a) = n\pi$, $n = 1, 2, 3, \ldots$ and the corresponding eigenfunctions are $\phi = \sin\left[n\pi\frac{\ln(r/a)}{\ln(b/a)}\right]$. The solution of the θ -equation satisfying $G(0) = 0$ is $G = \sinh\sqrt{\lambda}\theta = \sinh\frac{n\pi\theta}{\ln(b/a)}$. Thus by superposition

$$u = \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi\theta}{\ln(b/a)}\sin\left[n\pi\frac{\ln(r/a)}{\ln(b/a)}\right].$$

The nonhomogeneous boundary condition,

$$f(r) = \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi^2}{2\ln(b/a)}\sin\left[n\pi\frac{\ln(r/a)}{\ln(b/a)}\right],$$

will determine A_n. One method (for another, see exercise 5.3.9) is to let $z = \ln(r/a)/\ln(b/a)$. Then $a < r < b$, lets $0 < z < 1$. This is a sine series in z (with $L = 1$) and hence

$$A_n \sinh\frac{n\pi^2}{2\ln(b/a)} = 2\int_0^1 f(r)\sin\left[n\pi\frac{\ln(r/a)}{\ln(b/a)}\right] dz.$$

But $dz = dr/r\ln(b/a)$. Thus

$$A_n \sinh\frac{n\pi^2}{2\ln(b/a)} = \frac{2}{\ln(b/a)}\int_0^1 f(r)\sin\left[n\pi\frac{\ln(r/a)}{\ln(b/a)}\right] dr/r.$$

4. Solution is given by,

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} e^{-(n\pi/L)^2 kt}$$

Coefficients satisfy,

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}$$

$$A_0 = \frac{1}{L}\int_0^L f(x)\, dx$$

$$A_m = \frac{2}{L}\int_0^L f(x)\cos\frac{m\pi x}{L}\, dx$$

So, by inspection,

$A_0 = 6, A_3 = 4, A_{1,2,4,5,6,\ldots} = 0$

5.

Using one-dimensional eigenfunctions

$$u(x,y) = \sum_{n=1}^{\infty} a_n(y) \sin n\, x\,.$$

Substitute into the pde below,

$$\sum_{n=1}^{\infty} \left[\frac{d^2 b_n}{dy^2} - \left(\frac{n\pi}{L}\right)^2 b_n \right] \sin \frac{n\pi x}{L} = Q$$

which results in,

$$\sum_{n=1}^{\infty} \left[\frac{d^2 a_n}{dy^2} - n^2 a_n \right] \sin nx = e^{2y} \sin x\,.$$

Thus for $n \neq 1, a_n(y) = \alpha_n \sinh ny + \beta_n \cosh n\,y$. However, for $n = 1$, $\frac{d^2 a_1}{dy^2} - a_1 = e^{2y}$ and thus $a_1(y) = \frac{1}{3} e^{2y} + \alpha_1 \sinh y + \beta_1 \cosh y$, using the method of undetermined coefficients for a particular solution. The other boundary conditions are $a_n(0) = 0$ and $a_n(L) = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$. Thus $\frac{1}{3} + \beta_1 = 0$ and $(n \neq 1)\beta_n = 0$. Also $\frac{1}{3} e^{2L} + \alpha_1 \sinh L - \frac{1}{3} \cosh L = a_1(L)$ and $(n \neq 1)\alpha_n \sinh n\,L = a_n(L)$, determining all α_n.