MAK 501 HW 2 Solution

1) $f(x)$ is 12-periodic function. So, at the interval $[-8, 8]$

$$f(x) = \begin{cases} \frac{x}{2} + 2 & -6 < x < -2 \\ -\frac{x}{2} & -2 < x < 0 \\ -x & 0 < x < 2 \\ \frac{x}{2} - 3 & 2 < x < 4 \\ -1 & 4 < x < 6 \end{cases}$$

If f is a $2p$-periodic piecewise smooth function, the Fourier series of f is given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p} \right]$$

where

$$a_0 = \frac{1}{2p} \int_{-p}^{p} f(x)\, dx$$

$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi x}{p}\, dx \qquad (n = 1, 2, 3, \ldots)$$

$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi x}{p}\, dx \qquad (n = 1, 2, 3 \ldots)$$

$2p = 12 \rightarrow p = 6$

$$a_0 = \frac{1}{12} \left[\int_{-6}^{-2} \left[\frac{x}{2} + 2 \right] dx + \int_{-2}^{0} \left[-\frac{x}{2} \right] dx + \int_{0}^{2} \left[-x \right] dx + \int_{2}^{4} \left[\frac{x}{2} - 3 \right] dx + \int_{4}^{6} [-1]\, dx \right]$$

$$= \frac{1}{12} \left[\left\{ \frac{x^2}{4} + 2x \right\}_{-6}^{-2} + \left\{ -\frac{x^2}{4} \right\}_{-2}^{0} + \left\{ -\frac{x^2}{2} \right\}_{0}^{2} + \left\{ \frac{x^2}{4} - 3x \right\}_{2}^{4} + \left\{ -x \right\}_{4}^{6} \right]$$

$$a_0 = \frac{1}{12}\left\{ \left[1-4-3+12 \right] + \left[0+1 \right] + \left[-2-0 \right] + \left[4-12-1+6 \right] + \left[-6+4 \right] \right\}$$

$$= -\frac{1}{2}$$

$$a_n = \frac{1}{6}\left\{ \int_{-6}^{-2} \left[\frac{x}{2}+2 \right] \cos\frac{n\pi x}{6}\,dx + \int_{-2}^{0} \left[-\frac{x}{2} \right] \cos\frac{n\pi x}{6}\,dx + \int_{0}^{2} \left[-x \right] \cos\frac{n\pi x}{6}\,dx \right.$$

$$\left. + \int_{2}^{4} \left[\frac{x}{2}-3 \right] \cos\frac{n\pi x}{6}\,dx + \int_{4}^{6} \left[-1 \right] \cos\frac{n\pi x}{6}\,dx \right\}$$

from Asmar's useful integrals

$$\int x\cos ax\,dx = \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax$$

Then,

$$a_n = \frac{1}{6}\left\{ \frac{1}{2}\left[\frac{-12}{n\pi}\sin\left(-\frac{n\pi}{3}\right) + \frac{36}{n^2\pi^2}\cos\left(-\frac{n\pi}{3}\right) - \frac{-36}{n\pi}\sin(-n\pi) - \frac{36}{n^2\pi^2}\cos(-n\pi) \right] + \frac{12}{n\pi}\sin\left(\frac{n\pi}{3}\right) - \frac{12}{n\pi}\sin(n\pi) \right.$$

$$-\frac{1}{2}\left[0 + \frac{36}{n^2\pi^2} - \frac{-12}{n\pi}\sin\left(-\frac{n\pi}{3}\right) - \frac{36}{n^2\pi^2}\cos\left(-\frac{n\pi}{3}\right) \right]$$

$$-\left[\frac{12}{n\pi}\sin\left(\frac{n\pi}{3}\right) + \frac{36}{n^2\pi^2}\cos\left(\frac{n\pi}{3}\right) - 0 - \frac{36}{n^2\pi^2} \right]$$

$$+\frac{1}{2}\left[\frac{24}{n\pi}\sin\left(\frac{2n\pi}{3}\right) + \frac{36}{n^2\pi^2}\cos\left(\frac{2n\pi}{3}\right) - \frac{12}{n\pi}\sin\left(\frac{n\pi}{3}\right) - \frac{36}{n^2\pi^2}\cos\left(\frac{n\pi}{3}\right) \right] - \frac{18}{n\pi}\sin\left(\frac{2n\pi}{3}\right) + \frac{18}{n\pi}\sin\left(\frac{n\pi}{3}\right)$$

$$\left. -\frac{6}{n\pi}\sin(n\pi) + \frac{6}{n\pi}\sin\left(\frac{2n\pi}{3}\right) \right\}$$

By using $\sin(\alpha) = \sin(180-\alpha)$ & $\cos(\alpha) = -\cos(180-\alpha)$

$$a_n = \frac{3}{n^2\pi^2}\left[1 + (-1)^{n+1} - 2\cos\left(\frac{n\pi}{3}\right) \right]$$

$$b_n = \frac{1}{6}\left\{ \int_{-6}^{-2}\left[\frac{x}{2}+2\right]\sin\frac{n\pi x}{6} + \int_{-2}^{0}\left[-\frac{x}{2}\right]\sin\frac{n\pi x}{6}\,dx + \int_{0}^{2}\left[-x\right]\sin\frac{n\pi x}{6}\,dx\right.$$

$$\left. + \int_{2}^{4}\left[\frac{x}{2}-3\right]\sin\frac{n\pi x}{6}\,dx + \int_{4}^{6}\left[-1\right]\sin\frac{n\pi x}{6}\,dx\right\}$$

From Asmar's useful integrals

$$\int x\sin ax\,dx = -\frac{x}{a}\cos ax + \frac{1}{a^2}\sin ax$$

Then,

$$b_n = \frac{1}{6}\left\{\frac{1}{2}\left[\frac{12}{n\pi}\cos\left(-\frac{n\pi}{3}\right) + \frac{36}{n^2\pi^2}\sin\left(-\frac{n\pi}{3}\right) - \frac{36}{n\pi}\cos(-n\pi) + 0\right] - \frac{12}{n\pi}\cos\left(-\frac{n\pi}{3}\right) + \frac{12}{n\pi}\cos(-n\pi)\right.$$

$$-\frac{1}{2}\left[0 + 0 - \frac{12}{n\pi}\cos\left(-\frac{n\pi}{3}\right) - \frac{36}{n^2\pi^2}\sin\left(-\frac{n\pi}{3}\right)\right]$$

$$-\left[\frac{-12}{n\pi}\cos\left(\frac{n\pi}{3}\right) + \frac{36}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right) - 0 - 0\right]$$

$$+\frac{1}{2}\left[\frac{-24}{n\pi}\cos\left(\frac{2n\pi}{3}\right) + \frac{36}{n^2\pi^2}\sin\left(\frac{2n\pi}{3}\right) - \frac{-12}{n\pi}\cos\left(\frac{n\pi}{3}\right) - \frac{36}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right)\right] + \frac{18}{n\pi}\cos\left(\frac{2n\pi}{3}\right) - \frac{18}{n\pi}\cos\left(\frac{n\pi}{3}\right)$$

$$\left. + \frac{6}{n\pi}\cos(n\pi) - \frac{6}{n\pi}\cos\left(\frac{2n\pi}{3}\right)\right\}$$

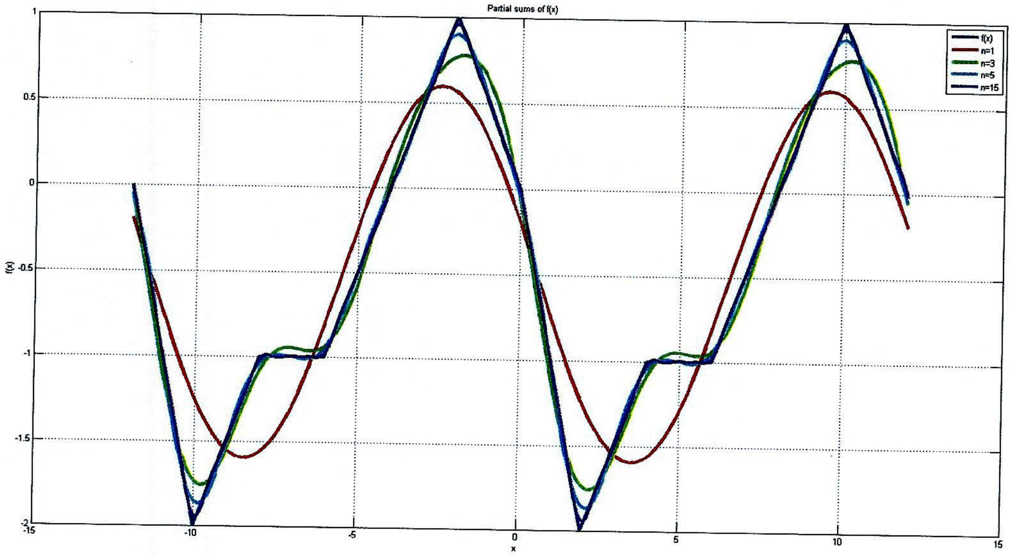
By using $\sin(\alpha) = \sin(180-\alpha)$ & $\cos(\alpha) = -\cos(180-\alpha)$

$$b_n = -\frac{12}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right)$$

$$f(x) = -\frac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{3}{n^2\pi^2}\left(1+(-1)^{n+1} - 2\cos\left(\frac{n\pi}{3}\right)\right)\right]\cos\frac{n\pi x}{6} - \frac{12}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right)\sin\frac{n\pi x}{6}$$

Fourier series converge to $f(x)$ at all x.

2)


Partial sums of f(x)
f(x)
n=1
n=3
n=5
n=15
1
0.5
0
-0.5
-1
-1.5
-2
f(x)
-15
-10
-5
0
5
10
15
x

3) Circle equation with center point $C(h,k)$ and radius r:

$$(x-h)^2 + (y-k)^2 = r^2$$

So, for the quarter in question: $(x+\pi)^2 + y^2 = \pi^2$

Thus:

$$f(x) = \begin{cases} \sqrt{\pi^2 - (x+\pi)^2} & -\pi \leq x \leq 0 \\ 0 & 0 \leq x \leq \pi \end{cases}$$

Fourier series of $2p$-periodic function:

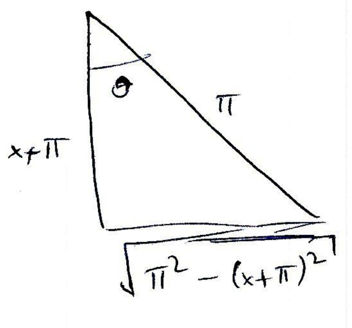
$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p} \right]$$

where

$$a_0 = \frac{1}{2p} \int_{-p}^{p} f(x)\, dx$$

$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi x}{p}\, dx \qquad (n = 1, 2, \ldots)$$

$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi x}{p}\, dx \qquad (n = 1, 2, \ldots)$$



$$x + \pi = \pi \cos\theta \quad \Rightarrow \quad dx = -\pi \sin\theta\, d\theta$$

$$x = \pi\cos\theta - \pi \qquad \theta = \cos^{-1}\left(\frac{x+\pi}{\pi}\right)$$

$$-\pi \leq x \leq 0 \quad \Rightarrow \quad -\frac{\pi}{2} \leq x \leq 0$$

$$2p = 2\pi \quad \Rightarrow \quad p = \pi$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi} \left[\int_{-\pi}^{0} \sqrt{\pi^2 - (x+\pi)^2}\, dx + \int_{0}^{\pi} 0\, dx \right]$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{0}\sqrt{\pi^2-(x+\pi)^2}\,dx = \frac{1}{2\pi}\int_{-\pi/2}^{0}\sqrt{\pi^2-\pi^2\cos^2\theta}\,(-\pi\sin\theta\,d\theta)$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{0}\pi\underbrace{|\sin\theta|}_{-\sin\theta}(-\pi\sin\theta)\,d\theta$$

$$= \frac{\pi}{2}\int_{-\pi/2}^{0}\sin^2\theta\,d\theta$$

$$= \frac{\pi}{2}\int_{-\pi/2}^{0}-\frac{1}{2}\left[\cos 2\theta - 1\right]d\theta$$

$$= -\frac{\pi}{4}\left[\frac{\sin 2\theta}{2}-\theta\right]\Bigg|_{-\pi/2}^{0}$$

$$= -\frac{\pi}{4}\left[0-0-0-\frac{\pi}{2}\right] = \frac{\pi^2}{8}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}\sqrt{\pi^2-(x+\pi)^2}\cos nx\,dx + \int_0^{\pi}0\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{0}\sqrt{\pi^2-\pi^2\cos^2\theta}\,\cos n(\pi\cos\theta-\pi)\left[-\pi\sin\theta\,d\theta\right]$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{0}\pi\underbrace{|\sin\theta|}_{-\sin\theta}\cos(n\pi\cos\theta)\underbrace{\cos n\pi}_{(-1)^n}\left[-\pi\sin\theta\,d\theta\right]$$

$$= \pi(-1)^n\int_{-\pi/2}^{0}\sin^2\theta\cos(n\pi\cos\theta)\,d\theta$$

$\cos(n\pi\cos\theta) = J_0 - 2J_2\cos 2\theta + 2J_4\cos 4\theta - \ldots$ $\quad J_0 = J_0(n\pi)$, $J_2 = J_2(n\pi)$, $\sin^2\theta = -\frac{1}{2}(\cos 2\theta - 1)$

$$a_n = (-1)^{n+1}\frac{\pi}{2}\int_{-\pi/2}^{0}(\cos 2\theta-1)(J_0-2J_2\cos 2\theta)\,d\theta = \frac{\pi}{2}(-1)^{n+1}\left[\int_{-\pi/2}^{0}J_0\cos 2\theta\,d\theta + \int_{-\pi/2}^{0}-2J_2\cos^2 2\theta\,d\theta\right.$$

$$\left.-\int_{-\pi/2}^{0}J_0\,d\theta + \int_{-\pi/2}^{0}2J_2\cos 2\theta\,d\theta\right] = \frac{\pi}{2}(-1)^{n+1}\left[\frac{1}{2}J_0\cancel{\sin 2\theta}^{\,0}\Big|_{-\pi/2}^{0} - J_2\left(\frac{1}{4}\cancel{\sin 4\theta}^{\,0}+\theta\right)\Big|_{-\pi/2}^{0} - J_0\theta\Big|_{-\pi/2}^{0}\right.$$

$$\left.+J_2\cancel{\sin 2\theta}^{\,0}\Big|_{-\pi/2}^{0}\right] = \frac{\pi}{2}(-1)^{n+1}\left[-J_2\frac{\pi}{2} - J_0\frac{\pi}{2}\right] = \frac{\pi^2}{4}(-1)^{n}\left[J_0+J_2\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}\sqrt{\pi^2-(x+\pi)^2}\,\sin nx\,dx + \int_{0}^{\pi} 0\,\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}\sqrt{\pi^2-(x+\pi)^2}\,\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{0}\sqrt{\pi^2-\pi^2\cos^2\theta}\,\sin(n\pi\cos\theta-\pi)\left[-\pi\sin\theta\,d\theta\right]$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{0}\pi\underbrace{|\sin\theta|}_{-\sin\theta}\left[\sin(n\pi\cos\theta)\underbrace{\cos n\pi}_{(-1)^n}\right]\left[-\pi\sin\theta\,d\theta\right]$$

$$= \pi(-1)^n\int_{-\pi/2}^{0}\sin^2\theta\,\sin(n\pi\cos\theta)\,d\theta$$

$$\sin(n\pi\cos\theta) = 2J_1\cos\theta - 2J_3\cos 3\theta + 2J_5\cos 5\theta \ldots \qquad J_1 = J_1(n\pi), \quad J_3 = J_3(n\pi)$$

$$\cancel{b_n = \frac{\pi}{2}(-1)^{n+1}\int_{-\pi/2}^{0}[J_1\cos}$$

$$b_n = \frac{\pi}{2}(-1)^{n+1}\int_{-\pi/2}^{0}(\cos 2\theta-1)(2J_1\cos\theta - 2J_3\cos 3\theta)\,d\theta$$

$$= \frac{\pi}{2}(-1)^{n+1}\int_{-\pi/2}^{0}\left[2J_1\cos\theta\cos 2\theta - 2J_3\cos 2\theta\cos 3\theta - 2J_1\cos\theta + 2J_3\cos 3\theta\right]d\theta$$

Note that $\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$

$$b_n = \frac{\pi}{2}(-1)^{n+1}\int_{-\pi/2}^{0}\left[J_1\cos 3\theta + J_1\cos\theta - J_3\cos 5\theta - J_3\cos\theta - 2J_1\cos\theta + 2J_3\cos 3\theta\right]d\theta$$

$$= \frac{\pi}{2}(-1)^{n+1}\left[\frac{1}{3}J_1\sin 3\theta - J_1\sin\theta - \frac{1}{5}J_3\sin 5\theta - J_3\sin\theta + \frac{2}{3}J_3\sin 3\theta\right]\Bigg|_{-\pi/2}^{0}$$

$$= \frac{\pi}{2}(-1)^{n+1}\Big[0-0-0-0+0-\frac{1}{3}J_1\underbrace{\sin\left(-\frac{3\pi}{2}\right)}_{+1} - J_1\underbrace{\sin\left(-\frac{\pi}{2}\right)}_{-1} - \frac{1}{5}J_3\underbrace{\sin\left(-\frac{5\pi}{2}\right)}_{-1}$$

$$- J_3\underbrace{\sin\left(-\frac{\pi}{2}\right)}_{-1} + \frac{2}{3}J_3\underbrace{\sin\left(-\frac{3\pi}{2}\right)}_{+1}\Big]$$

$$= \frac{\pi}{2}(-1)^{n+1}\left[\frac{2}{3}J_1 + \frac{28}{15}J_3\right]$$

$$f(x) = \frac{\pi^2}{8} + \sum_{n=1}^{\infty} \left\{ (-1)^n \frac{\pi^2}{4}\left[J_0 + J_2\right] \cos nx \right.$$

$$\left. + (-1)^{n+1} \frac{\pi}{2}\left[\frac{2}{3}J_1 + \frac{28}{15}J_3\right] \sin nx \right\}$$

4.

$$f(x) = \begin{cases} 1 & 0 < x < L/6 \\ 2 & L/6 < x < L/3 \\ 0 & L/3 < x < L \end{cases}$$

Fourier sine series of f(x):

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, 0 < x < L$$

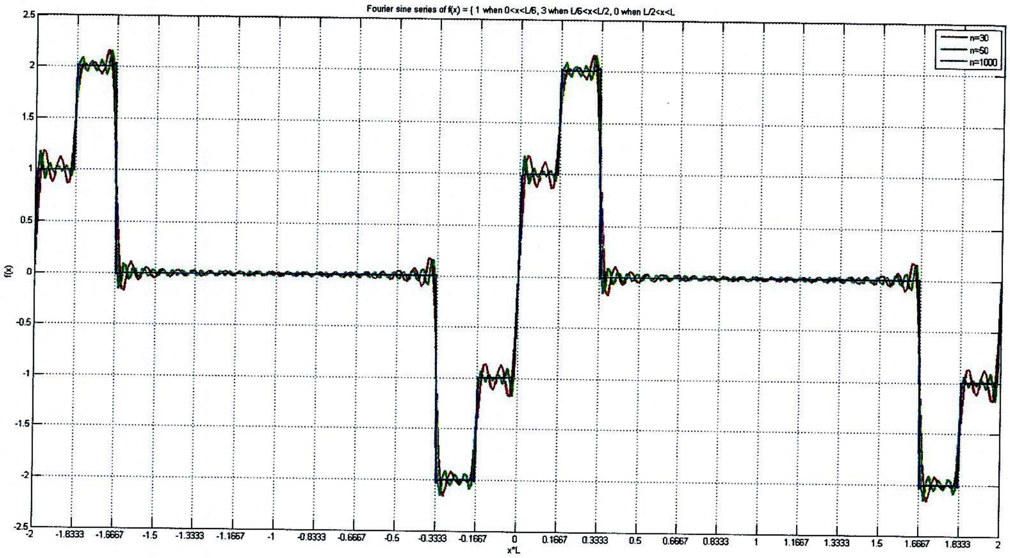
$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

$$b_n = \frac{2}{L} \left[\int_0^{L/6} \sin \frac{n\pi x}{L} dx + \int_{L/6}^{L/3} 2 \sin \frac{n\pi x}{L} dx + \int_{L/3}^{L} 0 \sin \frac{n\pi x}{L} dx \right]$$

$$b_n = -\frac{2}{n\pi} \left[\cos \frac{n\pi}{6} - 1 + 2 \cos \frac{n\pi}{3} - 2 \cos \frac{n\pi}{6} \right]$$

$$b_n = -\frac{2}{n\pi} \left[2 \cos \frac{n\pi}{3} - \cos \frac{n\pi}{6} - 1 \right]$$

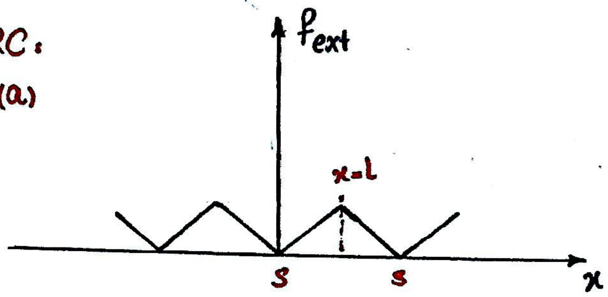
$$f(x) = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[\cos \frac{n\pi}{6} - 1 + 2 \cos \frac{n\pi}{3} - 2 \cos \frac{n\pi}{6} \right] \sin \frac{n\pi x}{L}$$



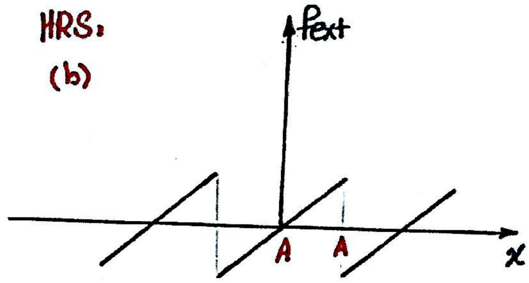
Fourier sine series of f(x) = { 1 when 0<x<L/6, 3 when L/6<x<L/2, 0 when L/2<x<L
n=30
n=50
n=1000
f(x)
x*L
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-2
-1.8333
-1.6667
-1.5
-1.3333
-1.1667
-1
-0.8333
-0.6667
-0.5
-0.3333
-0.1667
0
0.1667
0.3333
0.5
0.6667
0.8333
1
1.1667
1.3333
1.5
1.6667
1.8333
2

Ans. Q.5:

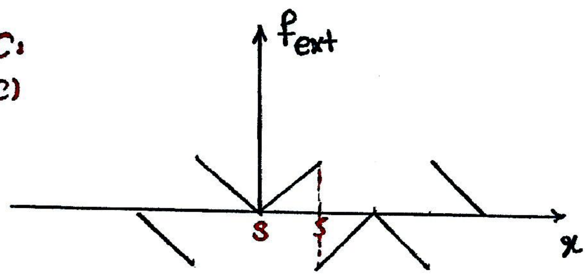
HRC:
(a)



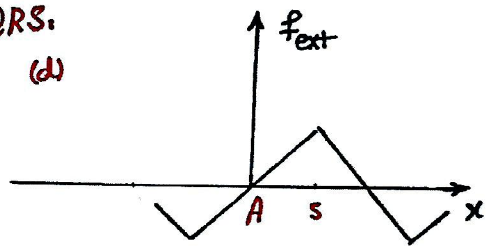
HRS:
(b)



QRC:
(c)



QRS:
(d)



From QRC graph, the period is $4L$ so $l = 2L$.

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$f_{ext}(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{2L} + b_n \sin\frac{n\pi x}{2L}\right)$$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx = \frac{1}{4L}\int_{-2L}^{2L} f_{ext}(x)\,dx$$

from the QRC graph, the $f_{ext}(x)$ is symmetric (even) about $x=0$ and antisymmeric about $x=L$

$$\text{so} \Rightarrow a_0 = \frac{1}{2L}\int_0^{2L} f_{ext}(x)\,dx$$

$$a_0 = \frac{1}{2L}\left[\int_0^{L} f_{ext}(x)\,dx + \int_L^{2L} f_{ext}(x)\,dx\right]$$

from graph. $\int_0^L f_{ext}(x)\,dx = -\int_L^{2L} f_{ext}(x)\,dx$

because $x=L$ is midpoint of the integration interval.

$$\Rightarrow \text{so} \quad \boxed{a_0 = 0}$$

$f_{ext}(x)$ is symmetric about $x=0$ and $\cos\left(\frac{n\pi x}{2L}\right)$ is also symmetric about that point. So, the product of them is symmetric about $x=0$.

$$a_n = \frac{1}{2L}\int_{-2L}^{2L} f_{ext}(x)\cos\frac{n\pi x}{2L}\,dx = \frac{2}{2L}\int_0^{2L} f_{ext}(x)\cos\frac{n\pi x}{2L}\,dx$$

$$a_n = \frac{1}{L}\int_0^{2L} f_{ext}(x)\cos\frac{n\pi x}{2L}\,dx \quad (*)$$

It is clear from QRC graph that $f_{ext}(x)$ is antisymmetric about $x=L$.

It can be clearly shown that $\cos\left(\frac{n\pi x}{2L}\right)$ is antisymmetric about that point if n is odd and symmetric about that point if n is even. Thus, the integrand in the integral (*) is symmetric about $x=L$ (midpoint of the integration interval) if n is odd, and anti symmetric about that point if n is even.

$$\text{Hence } a_n = \begin{cases} 0, & n \text{ even} \\ \frac{2}{L}\int_0^L f_{ext}(x)\cos\frac{n\pi x}{2L}\,dx, & n \text{ odd} \end{cases}$$

$$b_n = \frac{1}{2L}\int_{-2L}^{2L} f_{ext}(x)\sin\frac{n\pi x}{2L}\,dx$$

$f_{ext}(x)$ is symmetric and $\sin\left(\frac{n\pi x}{2L}\right)$ is antisymmetric about $x=0$. So, the integrand in the integral above is antisymmetric about that point. So, $\boxed{b_n=0}$

Hence:
$$\boxed{\begin{aligned} f(x) &= \sum_{n=1,3,\ldots}^{\infty} a_n\cos\frac{n\pi x}{2L}, \quad 0<x<L \\ a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{2L}\,dx \end{aligned}}$$

quarter-range cosine expansion

From the QRS graph, we see that $f_{ext}(x)$ is antisymmetric about $x=0$ and $\cos\left(\frac{n\pi x}{2L}\right)$ is symmetric about that point. Thus, the integrand in both following integrals will be antisymmetric about $x=0$. So, $\qquad l=2L$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx = \frac{1}{4L}\int_{-2L}^{2L} \underbrace{f_{ext}(x)}_{\text{antisymmetric}}\,dx = 0 \quad \Rightarrow \quad \boxed{a_0 = 0}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{1}{2L}\int_{-2L}^{2L} \underbrace{f_{ext}(x)\cos\frac{n\pi x}{2L}}_{\text{antisymmetric}}\,dx = 0 \quad \boxed{a_n = 0}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{2L}\int_{-2L}^{2L} f_{ext}(x)\sin\frac{n\pi x}{2L}\,dx = \frac{2}{2L}\int_{0}^{2L} f_{ext}(x)\sin\frac{n\pi x}{2L}\,dx$$

$$b_n = \frac{1}{L}\int_{0}^{2L} f_{ext}(x)\sin\frac{n\pi x}{2L}\,dx \quad (*)$$

It can be verified that $\sin\left(\frac{n\pi x}{2L}\right)$ is symmetric about $x=L$ if n is odd, and antisymmetric about that point if n is even. We see that $f_{ext}(x)$ is symmetric about $x=L$. Thus, the integrand in integral $(*)$ is symmetric about $x=L$ (midpoint of the integration interval) if n is odd, and antisymmetric about that point n is even.

$$b_n = \begin{cases} 0 & , n \text{ even} \\ \frac{2}{L}\int_{0}^{L} f_{ext}(x)\sin\frac{n\pi x}{2L} & , n \text{ odd} \end{cases}$$

Hence:

$$\boxed{\begin{aligned} f(x) &= \sum_{n=1,3,\dots}^{\infty} b_n \sin\frac{n\pi x}{2L}, \quad 0<x<L \\ b_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{2L}\,dx \end{aligned}}$$

quarter-range sine series