MAK 501

Homework 1 Solution

1.a

$$y' + 2y = x + e^{-3x} \tag{1}$$

$$y = y_h + y_p$$

y_h**: corresponding homogeneous D.E.:**

$$y' + 2y = 0$$

$$y' = -2y$$

Integrate

$$\ln y = -2x$$

$$y_h = Ce^{-2x} \tag{2}$$

y_p**: corresponding particular solution:**
for x, solution is in the form $y = A + Bx$ and for e^{-3x}, solution is in the form $y = De^{-3x}$.

$$y = A + Bx + De^{-3x} \tag{3}$$

Substitute 2 and 3 into 1

$$\underbrace{B - 3De^{-3x}}_{y'} + \underbrace{2(A + Bx + De^{-3x})}_{2y} = x + e^{-3x}$$

$A = -\frac{1}{4}$, $B = \frac{1}{2}$, $D = -1$

$$y = y_h + y_p = Ce^{-2x} - \frac{1}{4} + \frac{1}{2}x - e^{-3x}$$

To solve for C, initial value must be given.

1.b

When D.E. is in the form

$$y' + p(x)y = g(x)$$

The solution is found as

$$y = e^{-\int p(x)dx}\left[C + \int g(x)e^{\int p(x)dx}dx\right] = \frac{1}{\mu(x)}\left[C + \int g(x)\mu(x)dx\right]$$

where

$$\mu(x) = e^{\int p(x)dx}$$

is the integrating factor.

$$y' + \frac{1}{x}y = 3\sin x$$

$$\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$y = \frac{1}{x}\left[C + \int 3\sin(2x)xdx\right] = \frac{1}{x}\left[C + 3\int x\sin 2xdx\right]$$

where

$$\int x\sin 2xdx \underset{\substack{u=x \\ du=dx}}{} \underset{\substack{dv=\sin 2xdx \\ v=-\frac{1}{2}\cos 2x}}{=} [x]\left[-\frac{1}{2}\cos 2x\right] - \int -\frac{1}{2}\cos 2xdx = -\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x$$

$$y = \frac{1}{x}\left[C + 3\left\{-\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x\right\}\right]$$

$$y = \frac{C}{x} - \frac{3}{2}\cos 2x + \frac{3}{4x}\sin 2x$$

1.c

$$y' + \frac{2}{x}y = \frac{\sin x}{x^2}$$

Follow the procedure given in 1.b

$$\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$y = \frac{1}{x^2}\left[C + \int \frac{\sin x}{x^2}x^2 dx\right] = \frac{C}{x^2} - \frac{\cos x}{x^2}$$

Initial value:

$$y(\pi) = 0 = \frac{C}{\pi^2} - \frac{\cos \pi}{\pi^2}$$

$$y = -\frac{1}{x^2} - \frac{\cos x}{x^2} = -\frac{1 + \cos x}{x^2}$$

2.a

$$y' + \frac{1}{x}y = \cos x$$

Follow the procedure given in 1.b

$$\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$y = \frac{1}{x}\left[C + \int \cos x x dx\right]$$

where

$$\int x \cos x dx \underset{\substack{u=x \\ du=dx}}{} \underset{\substack{dv=\cos x dx \\ v=\sin x}}{=} x \sin x - \int \sin x dx = x \sin x + cosx$$

$$y = \frac{1}{x}\left[C + x \sin x + \cos x\right] = \frac{C}{x} + \sin x + \frac{\cos x}{x}$$

2.b

$$xy' + 2y = e^{3x}$$

By dividing both sides with x, obtain

$$y' + \frac{2}{x}y = \frac{e^{3x}}{x}$$

Follow the procedure given in 1.b

$$\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$y = \frac{1}{x^2}\left[C + \int \frac{e^{3x}}{x}x^2 dx\right] = \frac{1}{x^2}\left[C + \int xe^{3x} dx\right]$$

where

$$\int xe^{3x} dx \underset{\substack{u=x \\ du=dx}}{} \underset{\substack{dv=e^{3x}dx \\ v=\frac{1}{3}e^{3x}}}{=} x\frac{1}{3}e^{3x} - \int \frac{1}{3}e^{3x} dx = \frac{xe^{3x}}{3} - \frac{e^{3x}}{9}$$

$$y = \frac{1}{x^2}\left[C + \frac{xe^{3x}}{3} - \frac{e^{3x}}{9}\right]$$

$$y = \frac{C}{x^2} + \frac{e^{3x}}{3x} - \frac{e^{3x}}{9x^2}$$

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$$xy' + 2y = 3\sin x, \quad y(\pi) = \frac{1}{\pi}$$

Divide both sides with x to obtain

$$y' + \frac{2}{x}y = \frac{3}{x}\sin x$$

Follow the procedure given in 1.b

$$\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$y = \frac{1}{x^2}\left[C + \int \frac{3}{x}\sin(x)x^2 dx\right] = \frac{1}{x^2}\left[C + 3\int x\sin(x)dx\right]$$

where

$$\int x\sin(x)dx \underset{\substack{u=x \\ du=dx}}{} \underset{\substack{dv=\sin x dx \\ v=-\cos x}}{=} [x][-\cos x] - \int -\cos x dx = -x\cos x + \sin x$$

$$y = \frac{1}{x^2}[C + 3\{-x\cos x + \sin x\}]$$

$$y = \frac{C}{x^2} - \frac{3\cos x}{x} + \frac{3\sin x}{x^2}$$

$$y(\pi) = \frac{1}{\pi} = \frac{C}{\pi^2} - \frac{3\cos\pi}{\pi} + \frac{3\sin\pi}{\pi^2} \Rightarrow C = -2\pi$$

4.a

$$\frac{\delta u}{\delta t} - 5\frac{\delta u}{\delta x} = 0 \tag{4}$$

Linear change of variables:

$$\alpha = ax + bt$$

$$\beta = cx + dt$$

Chain rule in two dimensions:

$$\frac{\delta u}{\delta x} = \frac{\delta u}{\delta \alpha}\frac{\delta \alpha}{\delta x} + \frac{\delta u}{\delta \beta}\frac{\delta \beta}{\delta x} = a\frac{\delta u}{\delta \alpha} + c\frac{\delta u}{\delta \beta} \tag{5}$$

$$\frac{\delta u}{\delta t} = \frac{\delta u}{\delta \alpha}\frac{\delta \alpha}{\delta t} + \frac{\delta u}{\delta \beta}\frac{\delta \beta}{\delta t} = b\frac{\delta u}{\delta \alpha} + d\frac{\delta u}{\delta \beta} \tag{6}$$

Substitute 5 and 6 into 4

$$b\frac{\delta u}{\delta \alpha} + d\frac{\delta u}{\delta \beta} - 5\left[a\frac{\delta u}{\delta \alpha} + c\frac{\delta u}{\delta \beta}\right] = 0$$

Let $a = 1$, $b = 6$, $c = 1$, $d = 5$ to get rid of $\frac{\delta u}{\delta \beta}$.

$$\frac{\delta u}{\delta \alpha} = 0$$

$$u(\alpha, \beta) = f(\beta)$$

$$u(x, t) = f(x + 5t)$$

where f is any arbitrary function.

4.b

$$3\frac{\delta u}{\delta t} - \frac{\delta u}{\delta x} = 2 \tag{7}$$

Use the change of variables given in 4.a. Substitute 5 and 6 into 7

$$3\left[b\frac{\delta u}{\delta \alpha} + d\frac{\delta u}{\delta \beta}\right] - \left[a\frac{\delta u}{\delta \alpha} + c\frac{\delta u}{\delta \beta}\right] = 2$$

Let $a = 1$, $b = 1$, $c = 3$, $d = 1$ to get rid of $\frac{\delta u}{\delta \beta}$

$$2\frac{\delta u}{\delta \alpha} = 2$$

$$\frac{\delta u}{\delta \alpha} = 1$$

$$u(\alpha, \beta) = \alpha + f(\beta)$$

$$u(x, t) = x + t + f(3x + t)$$

where f is any arbitrary function.

5.a

Solution procedure is given below:

$$\frac{\delta u}{\delta x} + p(x,y)\frac{\delta u}{\delta y} = 0$$

$$\frac{dy}{dx} = p(x,y)$$

$$y = \int p(x,y)dx$$

$$C = y - \int p(x,y)dx = \Phi(x,y)$$

$$u(x,y) = f(y - \int p(x,y)dx)$$

So,

$$\frac{\delta u}{\delta x} + x^3\frac{\delta u}{\delta y} = 0$$

$$\frac{dy}{dx} = x^3 \Rightarrow y = \frac{x^4}{4} + C$$

$$C = y - \frac{x^4}{4} = \Phi(x,y)$$

$$u(x,y) = f(y - \frac{x^4}{4})$$

5.b

Solution procedure is given in 5.a

$$\frac{\delta u}{\delta x} + \cos x \frac{\delta u}{\delta y} = 0$$

$$\frac{dy}{dx} = \cos x \Rightarrow y = \sin x + C$$
$$C = y - \sin x = \Phi(x, y)$$
$$u(x, y) = f(y - \sin x)$$