

EXAMPLE 5.1

The stress distribution in a solid shaft has been plotted along three arbitrary radial lines as shown in Fig. 5–10*a*. Determine the resultant internal torque at the section.

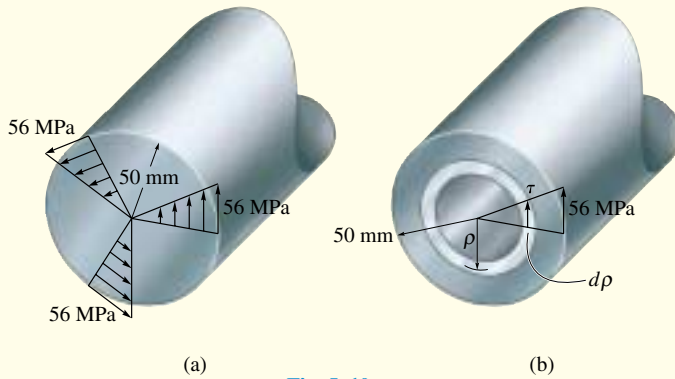


Fig. 5–10

Solution I

The polar moment of inertia for the cross-sectional area is

$$J = \frac{\pi}{2} (50 \text{ mm})^4 = 9.82 \times 10^6 \text{ mm}^4$$

Applying the torsion formula, with $\tau_{\max} = 56 \text{ MPa} = 56 \text{ N/mm}^2$, Fig. 5–10*a*, we have

$$\tau_{\max} = \frac{Tc}{J}; \quad 56 \text{ N/mm}^2 = \frac{T(50 \text{ mm})}{(9.82 \times 10^6) \text{ mm}^4}$$

$$T = 11.0 \text{ kN}\cdot\text{m} \quad \text{Ans.}$$

Solution II

The same result can be obtained by finding the torque produced by the stress distribution about the centroidal axis of the shaft. First we must express $\tau = f(\rho)$. Using proportional triangles, Fig. 5–10*b*, we have

$$\frac{\tau}{\rho} = \frac{56 \text{ N/mm}^2}{50 \text{ mm}}$$

$$\tau = 1.12\rho \text{ N/mm}^2$$

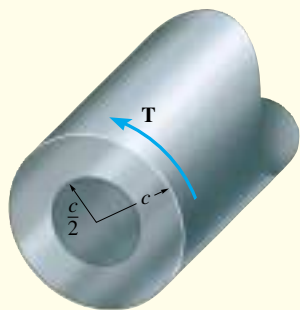
This stress acts on all portions of the differential ring element that has an area $dA = 2\pi\rho d\rho$. Since the force created by τ is $dF = \tau dA$, the torque is

$$dT = \rho dF = \rho(\pi dA) = \rho(1.12\rho)2\pi\rho d\rho = 2.24 \pi\rho^3 d\rho$$

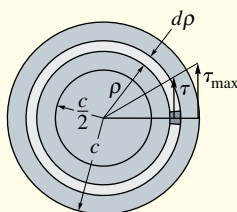
For the entire area over which τ acts, we require

$$T = \int_0^{50} 2.24 \pi\rho^3 d\rho = 2.24\pi \left(\frac{1}{4}\rho^4 \right) \Big|_0^{50} = 11.0 \times 10^6 \text{ N}\cdot\text{mm}$$

$$= 11.0 \text{ kN}\cdot\text{m} \quad \text{Ans.}$$



(a)



(b)

Fig. 5-11

The *solid* shaft of radius c is subjected to a torque \mathbf{T} , Fig. 5-11a. Determine the fraction of T that is resisted by the material contained within the outer region of the shaft, which has an inner radius of $c/2$ and outer radius c .

Solution

The stress in the shaft varies linearly, such that $\tau = (\rho/c)\tau_{max}$, Eq. 5-3. Therefore, the torque dT' on the ring (area) located within the lighter-shaded region, Fig. 5-11b, is

$$dT' = \rho(\tau dA) = \rho(\rho/c)\tau_{max}(2\pi\rho d\rho)$$

For the entire lighter-shaded area the torque is

$$\begin{aligned} T' &= \frac{2\pi\tau_{max}}{c} \int_{c/2}^c \rho^3 d\rho \\ &= \frac{2\pi\tau_{max}}{c} \frac{1}{4} \rho^4 \Big|_{c/2}^c \end{aligned}$$

So that

$$T' = \frac{15\pi}{32} \tau_{max} c^3 \tag{1}$$

This torque T' can be expressed in terms of the applied torque T by first using the torsion formula to determine the maximum stress in the shaft. We have

$$\tau_{max} = \frac{Tc}{J} = \frac{Tc}{(\pi/2)c^4}$$

or

$$\tau_{max} = \frac{2T}{\pi c^3}$$

Substituting this into Eq. 1 yields

$$T' = \frac{15}{16} T \tag{Ans.}$$

Here, approximately 94% of the torque is resisted by the lighter-shaded region, and the remaining 6% of T (or $\frac{1}{16}$) is resisted by the inner “core” of the shaft, $\rho = 0$ to $\rho = c/2$. As a result, the material located at the *outer region* of the shaft is highly effective in resisting torque, which justifies the use of tubular shafts as an efficient means for transmitting torque, and thereby saves material.

EXAMPLE 5.3

The shaft shown in Fig. 5–12*a* is supported by two bearings and is subjected to three torques. Determine the shear stress developed at points *A* and *B*, located at section *a–a* of the shaft, Fig. 5–12*b*.

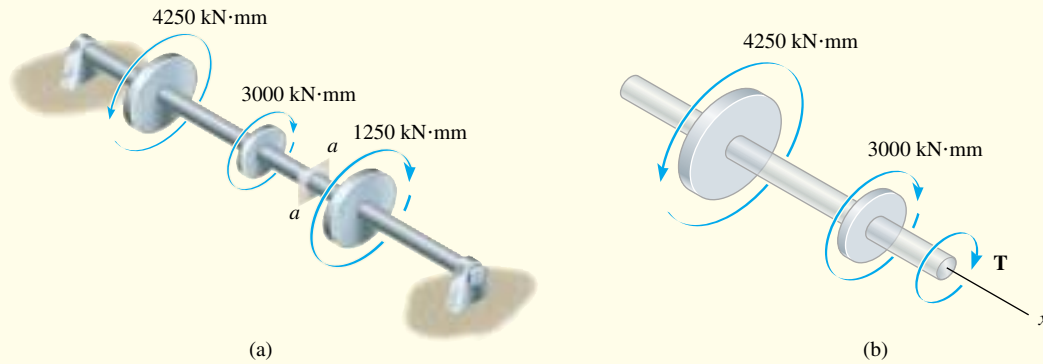


Fig. 5–12

Solution

Internal Torque. The bearing reactions on the shaft are zero, provided the shaft's weight is neglected. Furthermore, the applied torques satisfy moment equilibrium about the shaft's axis.

The internal torque at section *a–a* will be determined from the free-body diagram of the left segment, Fig. 5–12*b*. We have

$$\Sigma M_x = 0; \quad 4250 \text{ kN}\cdot\text{mm} - 3000 \text{ kN}\cdot\text{mm} - T = 0 \quad T = 1250 \text{ kN}\cdot\text{mm}$$

Section Property. The polar moment of inertia for the shaft is

$$J = \frac{\pi}{2} (75 \text{ mm})^4 = 4.97 \times 10^7 \text{ mm}^4$$

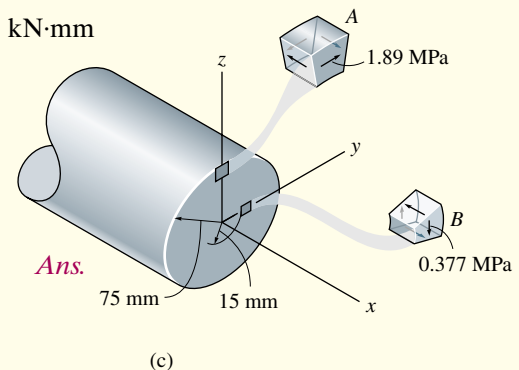
Shear Stress. Since point *A* is at $\rho = c = 75 \text{ mm}$,

$$\tau_B = \frac{Tc}{J} = \frac{1250 \text{ kN}\cdot\text{mm} \times 75 \text{ mm}}{4.97 \times 10^7 \text{ mm}^4} = 1.89 \text{ N/mm}^2 = 1.89 \text{ MPa} \quad \text{Ans.}$$

Likewise for point *B*, at $\rho = 15 \text{ mm}$, we have

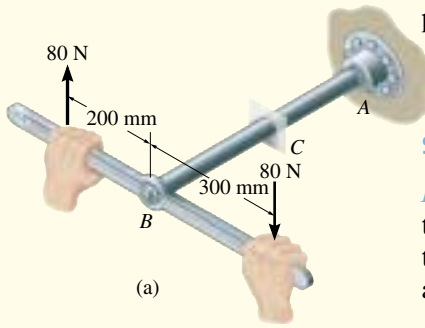
$$\tau_B = \frac{T\rho}{J} = \frac{1250 \text{ kN}\cdot\text{mm} \times 15 \text{ mm}}{4.97 \times 10^7 \text{ mm}^4} = 0.377 \text{ MPa} \quad \text{Ans.}$$

The directions of these stresses on each element at *A* and *B*, Fig. 5–12*c*, are established from the direction of the resultant internal torque **T**, shown in Fig. 5–12*b*. Note carefully how the shear stress acts on the planes of each of these elements.



EXAMPLE 5.4

The pipe shown in Fig. 5–13*a* has an inner diameter of 80 mm and an outer diameter of 100 mm. If its end is tightened against the support at *A* using a torque wrench at *B*, determine the shear stress developed in the material at the inner and outer walls along the central portion of the pipe when the 80-N forces are applied to the wrench.



Solution

Internal Torque. A section is taken at an intermediate location *C* along the pipe's axis, Fig. 5–13*b*. The only unknown at the section is the internal torque **T**. Force equilibrium and moment equilibrium about the *x* and *z* axes are satisfied. We require

$$\begin{aligned}\Sigma M_y = 0; \quad 80 \text{ N}(0.3 \text{ m}) + 80 \text{ N}(0.2 \text{ m}) - T &= 0 \\ T &= 40 \text{ N} \cdot \text{m}\end{aligned}$$

Section Property. The polar moment of inertia for the pipe's cross-sectional area is

$$J = \frac{\pi}{2} [(0.05 \text{ m})^4 - (0.04 \text{ m})^4] = 5.80(10^{-6}) \text{ m}^4$$

Shear Stress. For any point lying on the outside surface of the pipe, $\rho = c_o = 0.05 \text{ m}$, we have

$$\tau_o = \frac{Tc_o}{J} = \frac{40 \text{ N} \cdot \text{m}(0.05 \text{ m})}{5.80(10^{-6}) \text{ m}^4} = 0.345 \text{ MPa} \quad \text{Ans.}$$

And for any point located on the inside surface, $\rho = c_i = 0.04 \text{ m}$, so that

$$\tau_i = \frac{Tc_i}{J} = \frac{40 \text{ N} \cdot \text{m}(0.04 \text{ m})}{5.80(10^{-6}) \text{ m}^4} = 0.276 \text{ MPa} \quad \text{Ans.}$$

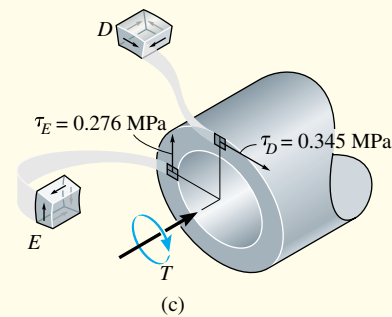
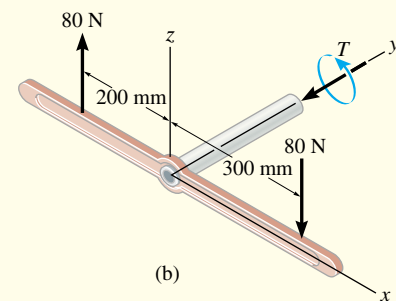


Fig. 5–13

To show how these stresses act at representative points *D* and *E* on the cross-sectional area, we will first view the cross section from the front of segment *CA* of the pipe, Fig. 5–13*a*. On this section, Fig. 5–13*c*, the resultant internal torque is equal but opposite to that shown in Fig. 5–13*b*. The shear stresses at *D* and *E* contribute to this torque and therefore act on the shaded faces of the elements in the directions shown. As a consequence, notice how the shear-stress components act on the other three faces. Furthermore, since the top face of *D* and the inner face of *E* are in stress-free regions taken from the pipe's outer and inner walls, no shear stress can exist on these faces or on the other corresponding faces of the elements.

EXAMPLE 5.5

A solid steel shaft AB shown in Fig. 5-14 is to be used to transmit 3750 W from the motor M to which it is attached. If the shaft rotates at $\omega = 175$ rpm and the steel has an allowable shear stress of $\tau_{\text{allow}} = 100$ MPa, determine the required diameter of the shaft to the nearest mm.

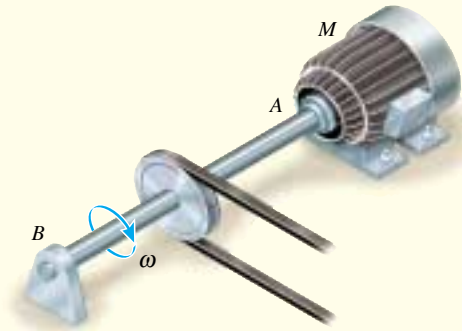


Fig. 5-14

Solution

The torque on the shaft is determined from Eq. 5-10, that is, $P = T\omega$. Expressing P in Newton-meters per second and ω in radians/second, we have

$$P = 3750 \text{ N}\cdot\text{m/s}$$

$$\omega = \frac{175 \text{ rev}}{\text{min}} \left(\frac{2\pi \text{ rad}}{1 \text{ rev}} \right) \left(\frac{1 \text{ min}}{60 \text{ s}} \right) = 18.33 \text{ rad/s}$$

Thus,

$$P = T\omega; \quad 3750 \text{ N}\cdot\text{m/s} = T(18.33) \text{ rad/s}$$

$$T = 204.6 \text{ N}\cdot\text{m}$$

Applying Eq. 5-12 yields

$$\begin{aligned} \frac{J}{c} &= \frac{\pi c^4}{2c} = \frac{T}{\tau_{\text{allow}}} \\ c &= \left(\frac{2T}{\pi\tau_{\text{allow}}} \right)^{1/3} = \left(\frac{2(204.6 \text{ N}\cdot\text{m})(1000 \text{ mm/m})}{\pi(100 \text{ N/mm}^2)} \right)^{1/3} \\ c &= 10.92 \text{ mm} \end{aligned}$$

Since $2c = 21.84$ mm, select a shaft having a diameter of

$$d = 22 \text{ mm}$$

Ans.

E X A M P L E 5.6

A tubular shaft, having an inner diameter of 30 mm and an outer diameter of 42 mm, is to be used to transmit 90 kW of power. Determine the frequency of rotation of the shaft so that the shear stress will not exceed 50 MPa.

Solution

The maximum torque that can be applied to the shaft is determined from the torsion formula.

$$\tau_{\max} = \frac{Tc}{J}$$
$$50(10^6) \text{ N/m}^2 = \frac{T(0.021 \text{ m})}{(\pi/2)[(0.021 \text{ m})^4 - (0.015 \text{ m})^4]}$$
$$T = 538 \text{ N} \cdot \text{m}$$

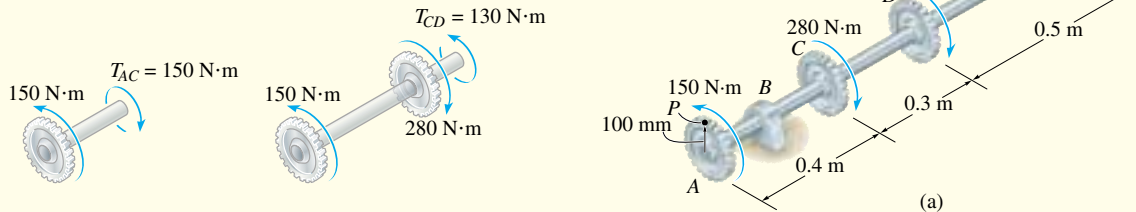
Applying Eq. 5-11, the frequency of rotation is

$$P = 2\pi fT$$
$$90(10^3) \text{ N} \cdot \text{m/s} = 2\pi f(538 \text{ N} \cdot \text{m})$$
$$f = 26.6 \text{ Hz}$$

Ans.

EXAMPLE 5.7

The gears attached to the fixed-end steel shaft are subjected to the torques shown in Fig. 5–20*a*. If the shear modulus of elasticity is 80 GPa and the shaft has a diameter of 14 mm, determine the displacement of the tooth *P* on gear *A*. The shaft turns freely within the bearing at *B*.



Solution

Internal Torque. By inspection, the torques in segments *AC*, *CD*, and *DE* are different yet *constant* throughout each segment. Free-body diagrams of appropriate segments of the shaft along with the calculated internal torques are shown in Fig. 5–20*b*. Using the right-hand rule and the established sign convention that positive torque is directed away from the sectioned end of the shaft, we have

$$T_{AC} = +150 \text{ N}\cdot\text{m} \quad T_{CD} = -130 \text{ N}\cdot\text{m} \quad T_{DE} = -170 \text{ N}\cdot\text{m}$$

These results are also shown on the torque diagram, Fig. 5–20*c*.

Angle of Twist. The polar moment of inertia for the shaft is

$$J = \frac{\pi}{2}(0.007 \text{ m})^4 = 3.77(10^{-9}) \text{ m}^4$$

Applying Eq. 5–16 to each segment and adding the results algebraically, we have

$$\begin{aligned} \phi_A = \sum \frac{TL}{JG} &= \frac{(+150 \text{ N}\cdot\text{m})(0.4 \text{ m})}{3.77(10^{-9}) \text{ m}^4 [80(10^9) \text{ N/m}^2]} \\ &+ \frac{(-130 \text{ N}\cdot\text{m})(0.3 \text{ m})}{3.77(10^{-9}) \text{ m}^4 [80(10^9) \text{ N/m}^2]} \\ &+ \frac{(-170 \text{ N}\cdot\text{m})(0.5 \text{ m})}{3.77(10^{-9}) \text{ m}^4 [80(10^9) \text{ N/m}^2]} = -0.212 \text{ rad} \end{aligned}$$

Since the answer is negative, by the right-hand rule the thumb is directed *toward* the end *E* of the shaft, and therefore gear *A* will rotate as shown in Fig. 5–20*d*.

The displacement of tooth *P* on gear *A* is

$$s_P = \phi_A r = (0.212 \text{ rad})(100 \text{ mm}) = 21.2 \text{ mm} \quad \text{Ans.}$$

Remember that this analysis is valid only if the shear stress does not exceed the proportional limit of the material.

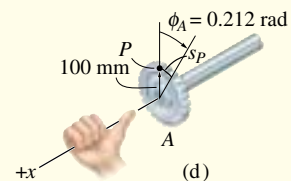
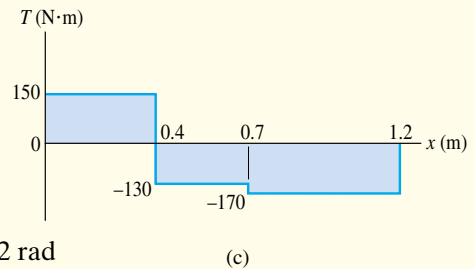
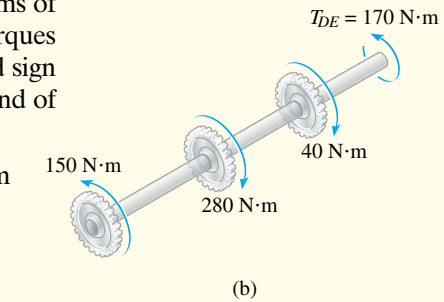
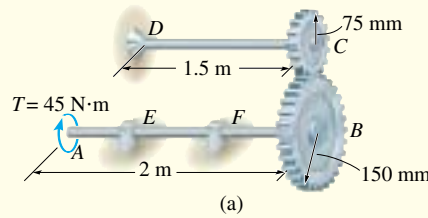


Fig. 5–20

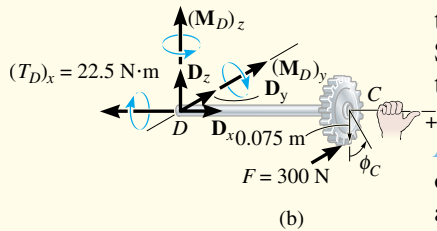
EXAMPLE 5.8

The two solid steel shafts shown in Fig. 5–21a are coupled together using the meshed gears. Determine the angle of twist of end A of shaft AB when the torque $T = 45 \text{ N} \cdot \text{m}$ is applied. Take $G = 80 \text{ GPa}$. Shaft AB is free to rotate within bearings E and F, whereas shaft DC is fixed at D. Each shaft has a diameter of 20 mm.



Solution

Internal Torque. Free-body diagrams for each shaft are shown in Fig. 5–21b and 5–21c. Summing moments along the x axis of shaft AB yields the tangential reaction between the gears of $F = 45 \text{ N} \cdot \text{m} / 0.15 \text{ m} = 300 \text{ N}$. Summing moments about the x axis of shaft DC, this force then creates a torque of $(T_D)_x = 300 \text{ N}(0.075 \text{ m}) = 22.5 \text{ N} \cdot \text{m}$ on shaft DC.

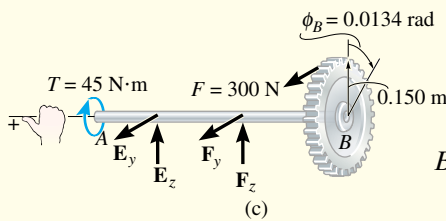


Angle of Twist. To solve the problem, we will first calculate the rotation of gear C due to the torque of $22.5 \text{ N} \cdot \text{m}$ in shaft DC, Fig. 5–21b. This angle of twist is

$$\phi_C = \frac{TL_{DC}}{JG} = \frac{(+22.5 \text{ N} \cdot \text{m})(1.5 \text{ m})}{(\pi/2)(0.010 \text{ m})^4[80(10^9) \text{ N/m}^2]} = +0.0269 \text{ rad}$$

Since the gears at the end of the shaft are in mesh, the rotation ϕ_C of gear C causes gear B to rotate ϕ_B , Fig. 5–21c, where

$$\begin{aligned} \phi_B(0.15 \text{ m}) &= (0.0269 \text{ rad})(0.075 \text{ m}) \\ \phi_B &= 0.0134 \text{ rad} \end{aligned}$$



We will now determine the angle of twist of end A with respect to end B of shaft AB caused by the $45 \text{ N} \cdot \text{m}$ torque, Fig. 5–21c. We have

$$\phi_{A/B} = \frac{T_{AB}L_{AB}}{JG} = \frac{(+45 \text{ N} \cdot \text{m})(2 \text{ m})}{(\pi/2)(0.010 \text{ m})^4[80(10^9) \text{ N/m}^2]} = +0.0716 \text{ rad}$$

The rotation of end A is therefore determined by adding ϕ_B and $\phi_{A/B}$, since both angles are in the same direction, Fig. 5–21c. We have

$$\phi_A = \phi_B + \phi_{A/B} = 0.0134 \text{ rad} + 0.0716 \text{ rad} = +0.0850 \text{ rad} \text{ Ans.}$$

Fig. 5–21

EXAMPLE 5.9

The 50-mm-diameter solid cast-iron post shown in Fig. 5-22*a* is buried 600 mm in soil. If a torque is applied to its top using a rigid wrench, determine the maximum shear stress in the post and the angle of twist at its top. Assume that the torque is about to turn the post, and the soil exerts a uniform torsional resistance of t N·mm/mm along its 600 mm buried length. $G = 40(10^3)$ MPa.

Solution

Internal Torque. The internal torque in segment AB of the post is constant. From the free-body diagram, Fig. 5-22*b*, we have

$$\Sigma M_z = 0; \quad T_{AB} = 100 \text{ N}(300 \text{ mm}) = 30 \times 10^3 \text{ N}\cdot\text{mm}$$

The magnitude of the uniform distribution of torque along the buried segment BC can be determined from equilibrium of the entire post, Fig. 5-22*c*. Here

$$\Sigma M_z = 0 \quad 100 \text{ N}(300 \text{ mm}) - t(600 \text{ mm}) = 0$$

$$t = 50 \text{ N}\cdot\text{mm}$$

Hence, from a free-body diagram of a section of the post located at the position x within region BC , Fig. 5-22*d*, we have

$$\Sigma M_z = 0; \quad T_{BC} - 50x = 0$$

$$T_{BC} = 50x$$

Maximum Shear Stress. The largest shear stress occurs in region AB , since the torque is largest there and J is constant for the post. Applying the torsion formula, we have

$$\tau_{\max} = \frac{T_{AB}^C}{J} = \frac{30 \times 10^3 \text{ N}\cdot\text{mm} (25 \text{ mm})}{(\pi/2)(25 \text{ mm})^4} = 1.22 \text{ N/mm}^2 \quad \text{Ans.}$$

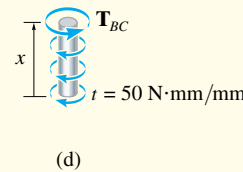
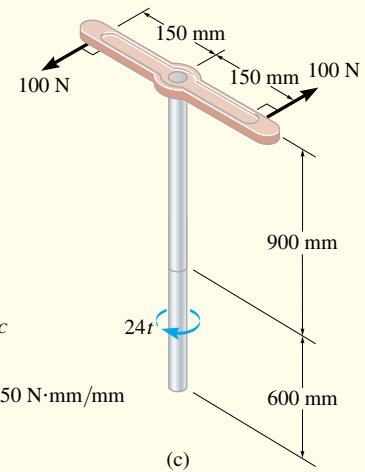
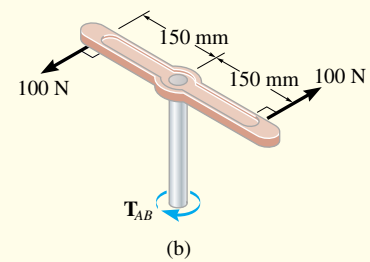
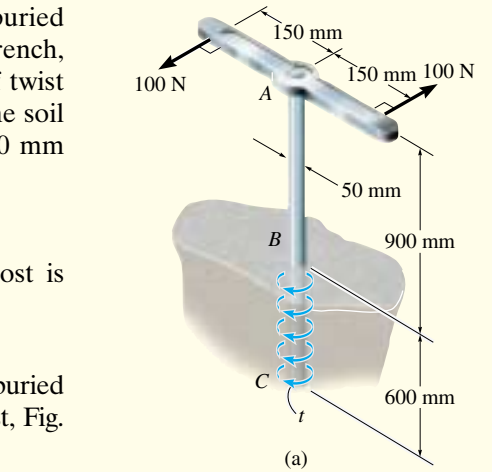
Angle of Twist. The angle of twist at the top can be determined relative to the bottom of the post, since it is fixed and yet is about to turn. Both segments AB and BC twist, and so in this case we have

$$\phi_A = \frac{T_{AB}L_{AB}}{JG} + \int_0^{L_{BC}} \frac{T_{BC} dx}{JG}$$

$$= \frac{(30 \times 10^3 \text{ N}\cdot\text{mm})(900 \text{ mm})}{JG} + \int_0^{600} \frac{50x dx}{JG}$$

$$= \frac{27 \times 10^6 \text{ N}\cdot\text{mm}^2}{JG} + \frac{50[(600)^2/2] \text{ N}\cdot\text{mm}^2}{JG}$$

$$= \frac{30 \times 10^6 \text{ N}\cdot\text{mm}^2}{(\pi/2)(25 \text{ mm})^4 40(10^3) \text{ N}\cdot\text{mm}^2} = 0.00147 \text{ rad}$$



Ans.

Fig. 5-22

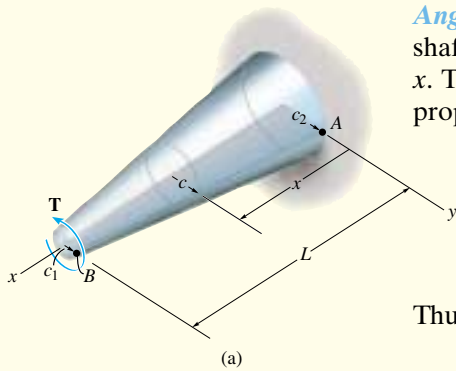
EXAMPLE 5.10

The tapered shaft shown in Fig. 5–23a is made of a material having a shear modulus G . Determine the angle of twist of its end B when subjected to the torque.

Solution

Internal Torque. By inspection or from the free-body diagram of a section located at the arbitrary position x , Fig. 5–23b, the internal torque is T .

Angle of Twist. Here the polar moment of inertia varies along the shaft's axis and therefore we must express it in terms of the coordinate x . The radius c of the shaft at x can be determined in terms of x by proportion of the slope of line AB in Fig. 5–23c. We have



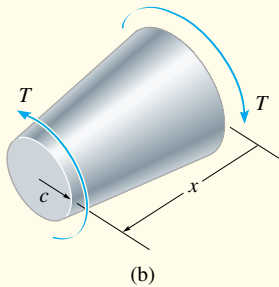
$$\frac{c_2 - c_1}{L} = \frac{c_2 - c}{x}$$

$$c = c_2 - x\left(\frac{c_2 - c_1}{L}\right)$$

Thus, at x ,

$$J(x) = \frac{\pi}{2} \left[c_2 - x\left(\frac{c_2 - c_1}{L}\right) \right]^4$$

Applying Eq. 5–14, we have



$$\phi = \int_0^L \frac{T dx}{\left(\frac{\pi}{2}\right) \left[c_2 - x\left(\frac{c_2 - c_1}{L}\right) \right]^4 G} = \frac{2T}{\pi G} \int_0^L \frac{dx}{\left[c_2 - x\left(\frac{c_2 - c_1}{L}\right) \right]^4}$$

Performing the integration using an integral table, the result becomes

$$\phi = \left(\frac{2T}{\pi G}\right) \frac{1}{3\left(\frac{c_2 - c_1}{L}\right) \left[c_2 - x\left(\frac{c_2 - c_1}{L}\right) \right]^3} \Bigg|_0^L$$

$$= \frac{2T}{\pi G} \left(\frac{L}{3(c_2 - c_1)}\right) \left(\frac{1}{c_1^3} - \frac{1}{c_2^3}\right)$$

Rearranging terms yields

$$\phi = \frac{2TL}{3\pi G} \left(\frac{c_2^2 + c_1c_2 + c_1^2}{c_1^3c_2^3}\right) \quad \text{Ans.}$$

To partially check this result, note that when $c_1 = c_2 = c$, then

$$\phi = \frac{TL}{[(\pi/2)c^4]G} = \frac{TL}{JG}$$

which is Eq. 5–15.

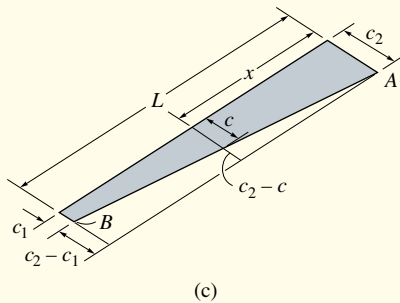
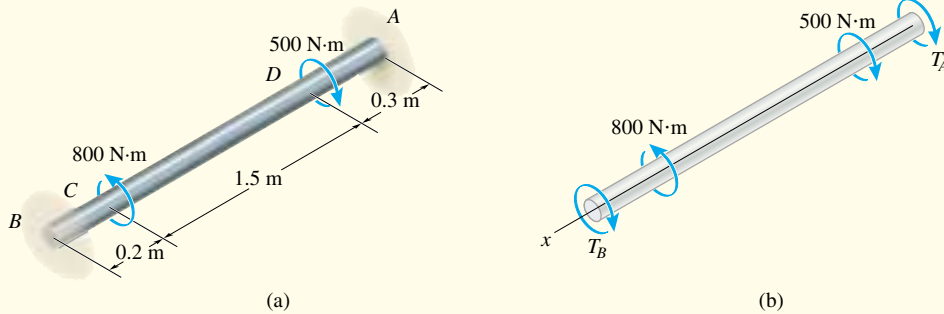


Fig. 5–23

EXAMPLE 5.11

The solid steel shaft shown in Fig. 5–25a has a diameter of 20 mm. If it is subjected to the two torques, determine the reactions at the fixed supports *A* and *B*.



Solution

Equilibrium. By inspection of the free-body diagram, Fig. 5–25b, it is seen that the problem is statically indeterminate since there is only *one* available equation of equilibrium, whereas \mathbf{T}_A and \mathbf{T}_B are unknown. We require

$$\Sigma M_x = 0; \quad -T_B + 800 \text{ N}\cdot\text{m} - 500 \text{ N}\cdot\text{m} - T_A = 0 \quad (1)$$

Compatibility. Since the ends of the shaft are fixed, the angle of twist of one end of the shaft with respect to the other must be zero. Hence, the compatibility equation can be written as

$$\phi_{A/B} = 0$$

This condition can be expressed in terms of the unknown torques by using the load–displacement relationship, $\phi = TL/JG$. Here there are three regions of the shaft where the internal torque is constant, BC, CD, and DA. On the free-body diagrams in Fig. 5–25c we have shown the internal torques acting on segments of the shaft which are sectioned in each of these regions. Using the sign convention established in Sec. 5.4, we have

$$\frac{-T_B(0.2 \text{ m})}{JG} + \frac{(T_A + 500 \text{ N}\cdot\text{m})(1.5 \text{ m})}{JG} + \frac{T_A(0.3 \text{ m})}{JG} = 0$$

or

$$1.8T_A - 0.2T_B = -750 \quad (2)$$

Solving Eqs. 1 and 2 yields

$$T_A = -345 \text{ N}\cdot\text{m} \quad T_B = 645 \text{ N}\cdot\text{m} \quad \text{Ans.}$$

The negative sign indicates that \mathbf{T}_A acts in the opposite direction of that shown in Fig. 5–25b.

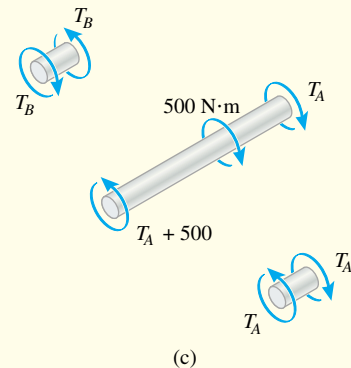


Fig. 5–25

E X A M P L E 5.12

The shaft shown in Fig. 5–26*a* is made from a steel tube, which is bonded to a brass core. If a torque of $T = 250 \text{ N}\cdot\text{m}$ is applied at its end, plot the shear-stress distribution along a radial line of its cross-sectional area. Take $G_{\text{st}} = 80 \text{ GPa}$, $G_{\text{br}} = 36 \text{ GPa}$.

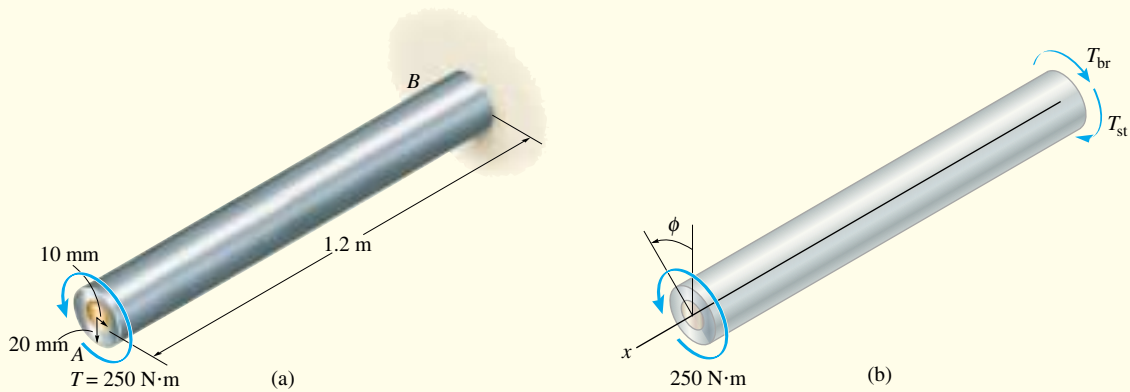


Fig. 5–26

Solution

Equilibrium. A free-body diagram of the shaft is shown in Fig. 5–26*b*. The reaction at the wall has been represented by the unknown amount of torque resisted by the steel, T_{st} , and by the brass, T_{br} . Equilibrium requires

$$-T_{\text{st}} - T_{\text{br}} + 250 \text{ N}\cdot\text{m} = 0 \quad (1)$$

Compatibility. We require the angle of twist of end A to be the same for both the steel and brass since they are bonded together. Thus,

$$\phi = \phi_{\text{st}} = \phi_{\text{br}}$$

Applying the load–displacement relationship, $\phi = TL/JG$, we have

$$\frac{T_{\text{st}}L}{(\pi/2)[(20 \text{ mm})^4 - (10 \text{ mm})^4]80(10^3) \text{ N/mm}^2} = \frac{T_{\text{br}}L}{(\pi/2)(10 \text{ mm})^4 36(10^3) \text{ N/mm}^2}$$

$$T_{\text{st}} = 33.33 T_{\text{br}} \quad (2)$$

Solving Eqs. 1 and 2, we get

$$T_{st} = 242.72 \text{ N} \cdot \text{m}$$

$$T_{br} = 7.28 \text{ N} \cdot \text{m}$$

These torques act throughout the entire length of the shaft, since no external torques act at intermediate points along the shaft's axis. The shear stress in the brass core varies from zero at its center to a maximum at the interface where it contacts the steel tube. Using the torsion formula,

$$(\tau_{br})_{\max} = \frac{7.28 \text{ N} \cdot \text{m} \cdot (10^3) \text{ mm/m} \cdot 10 \text{ mm}}{(\pi/2)(10 \text{ mm})^4} = 4.63 \text{ N/mm}^2 = 4.63 \text{ MPa}$$

For the steel, the minimum shear stress is also at this interface,

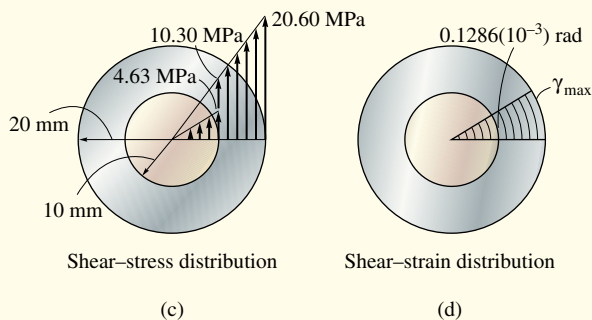
$$(\tau_{st})_{\min} = \frac{242.72 \text{ N} \cdot \text{m} \cdot 10^3 \text{ mm/m} \cdot 10 \text{ mm}}{(\pi/2)[(20 \text{ mm})^4 - (10 \text{ mm})^4]} = 10.30 \text{ N/mm}^2 = 10.30 \text{ MPa}$$

and the maximum shear stress is at the outer surface,

$$(\tau_{st})_{\max} = \frac{242.72 \text{ N} \cdot \text{m} \cdot 10^3 \text{ mm/m} \cdot 20 \text{ mm}}{(\pi/2)[(20 \text{ mm})^4 - (10 \text{ mm})^4]} = 20.60 \text{ N/mm}^2 = 20.60 \text{ MPa}$$

The results are plotted in Fig. 5–26c. Note the discontinuity of *shear stress* at the brass and steel interface. This is to be expected, since the materials have different moduli of rigidity; i.e., steel is stiffer than brass ($G_{st} > G_{br}$) and thus it carries more shear stress at the interface. Although the shear stress is discontinuous here, the *shear strain* is not. Rather, the shear strain is the *same* for both the brass and the steel. This can be shown by using Hooke's law, $\gamma = \tau/G$. At the interface, Fig. 5–26d, the shear strain is

$$\gamma = \frac{\tau}{G} = \frac{4.63 \text{ N/mm}^2}{36(10^3) \text{ N/mm}^2} = 0.1286(10^{-3}) \text{ rad}$$



EXAMPLE 5.13

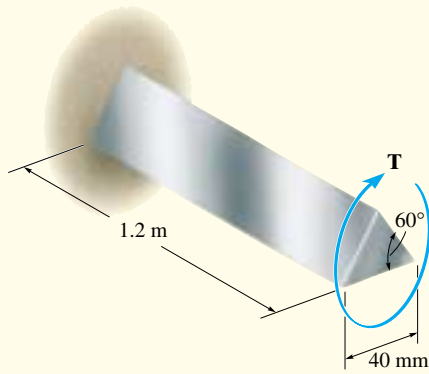


Fig. 5-29

The 6061-T6 aluminum shaft shown in Fig. 5-29 has a cross-sectional area in the shape of an equilateral triangle. Determine the largest torque \mathbf{T} that can be applied to the end of the shaft if the allowable shear stress is $\tau_{\text{allow}} = 56 \text{ MPa}$ and the angle of twist at its end is restricted to $\phi_{\text{allow}} = 0.02 \text{ rad}$. How much torque can be applied to a shaft of circular cross section made from the same amount of material? $G_{al} = 26 \text{ GPa}$.

Solution

By inspection, the resultant internal torque at any cross section along the shaft's axis is also \mathbf{T} . Using the formulas for τ_{max} and ϕ in Table 5-1, we require

$$\tau_{\text{allow}} = \frac{20T}{a^3}; \quad 56 \text{ N/mm}^2 = \frac{20T}{(40 \text{ mm})^3}$$

$$T = 179.2(10^3) \text{ N}\cdot\text{mm} = 179.2 \text{ N}\cdot\text{m}$$

Also,

$$\phi_{\text{allow}} = \frac{46TL}{a^4 G_{al}}; \quad 0.02 \text{ rad} = \frac{46T(1.2 \text{ m})(10^3) \text{ mm/m}}{(40 \text{ mm})^4 [26(10^3) \text{ N/mm}^2]}$$

$$T = 24.12(10^3) \text{ N}\cdot\text{mm} = 24.12 \text{ N}\cdot\text{m} \quad \text{Ans.}$$

By comparison, the torque is limited due to the angle of twist.

Circular Cross Section. If the same amount of aluminum is to be used in making the same length of shaft having a circular cross section, then the radius of the cross section can be calculated. We have

$$A_{\text{circle}} = A_{\text{triangle}}; \quad \pi c^2 = \frac{1}{2} (40 \text{ mm}) (40 \sin 60^\circ)$$

$$c = 14.850 \text{ mm}$$

The limitations of stress and angle of twist then require

$$\tau_{\text{allow}} = \frac{Tc}{J}; \quad 56 \text{ N/mm}^2 = \frac{T(14.850 \text{ mm})}{(\pi/2)(14.850 \text{ mm})^4}$$

$$T = 288.06(10^3) \text{ N}\cdot\text{mm} = 288.06 \text{ N}\cdot\text{m}$$

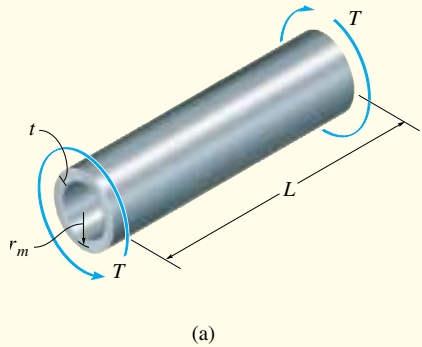
$$\phi_{\text{allow}} = \frac{TL}{JG_{al}}; \quad 0.02 \text{ rad} = \frac{T(1.2 \text{ m})(10^3) \text{ mm/m}}{(\pi/2)(14.85 \text{ mm})^4 [26(10^3) \text{ N/mm}^2]}$$

$$T = 33.10(10^3) \text{ N}\cdot\text{mm} = 33.10 \text{ N}\cdot\text{m} \quad \text{Ans.}$$

Again, the angle of twist limits the applied torque.

Comparing this result (33.10 N·m) with that given above (24.12 N·m), it is seen that a shaft of circular cross section can support 37% more torque than the one having a triangular cross section.

EXAMPLE 5.14



Calculate the average shear stress in a thin-walled tube having a circular cross section of mean radius r_m and thickness t , which is subjected to a torque T , Fig. 5–31a. Also, what is the relative angle of twist if the tube has a length L ?

Solution

Average Shear Stress. The mean area for the tube is $A_m = \pi r_m^2$. Applying Eq. 5–18 gives

$$\tau_{\text{avg}} = \frac{T}{2tA_m} = \frac{T}{2\pi t r_m^2} \quad \text{Ans.}$$

We can check the validity of this result by applying the torsion formula. In this case, using Eq. 5–9, we have

$$\begin{aligned} J &= \frac{\pi}{2}(r_o^4 - r_i^4) \\ &= \frac{\pi}{2}(r_o^2 + r_i^2)(r_o - r_i) \\ &= \frac{\pi}{2}(r_o^2 + r_i^2)(r_o + r_i)(r_o - r_i) \end{aligned}$$

Since $r_m \approx r_o \approx r_i$ and $t = r_o - r_i$, $J = \frac{\pi}{2}(2r_m^2)(2r_m)t = 2\pi r_m^3 t$

so that
$$\tau_{\text{avg}} = \frac{T r_m}{J} = \frac{T r_m}{2\pi r_m^3 t} = \frac{T}{2\pi t r_m^2} \quad \text{Ans.}$$

which agrees with the previous result.

The average shear-stress distribution acting throughout the tube's cross section is shown in Fig. 5–31b. Also shown is the shear-stress distribution acting on a radial line as calculated using the torsion formula. Notice how each τ_{avg} acts in a direction such that it contributes to the resultant torque \mathbf{T} at the section. As the tube's thickness decreases, the shear stress throughout the tube becomes more uniform.

Angle of Twist. Applying Eq. 5–20, we have

$$\phi = \frac{TL}{4A_m^2 G} \oint \frac{ds}{t} = \frac{TL}{4(\pi r_m^2)^2 G t} \oint ds$$

The integral represents the length around the centerline boundary, which is $2\pi r_m$. Substituting, the final result is

$$\phi = \frac{TL}{2\pi r_m^3 G t} \quad \text{Ans.}$$

Show that one obtains this same result using Eq. 5–15.

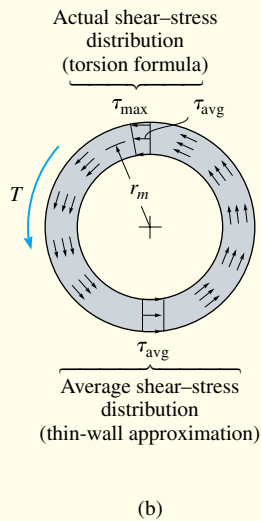
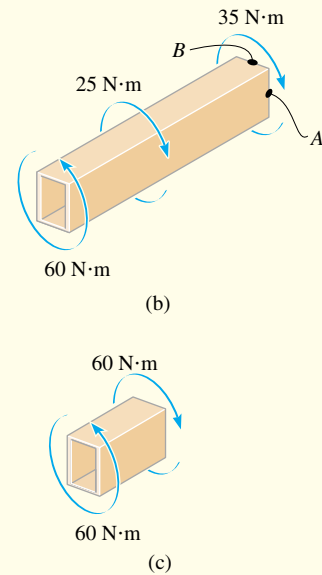
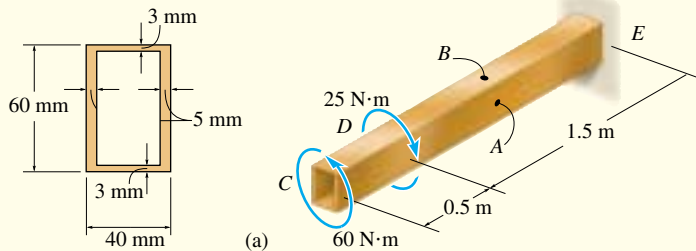


Fig. 5–31

EXAMPLE 5.15

The tube is made of C86100 bronze and has a rectangular cross section as shown in Fig. 5–32*a*. If it is subjected to the two torques, determine the average shear stress in the tube at points *A* and *B*. Also, what is the angle of twist of end *C*? The tube is fixed at *E*.



Solution

Average Shear Stress. If the tube is sectioned through points *A* and *B*, the resulting free-body diagram is shown in Fig. 5–32*b*. The internal torque is $35 \text{ N}\cdot\text{m}$. As shown in Fig. 5–32*d*, the area A_m is

$$A_m = (0.035 \text{ m})(0.057 \text{ m}) = 0.00200 \text{ m}^2$$

Applying Eq. 5–18 for point *A*, $t_A = 5 \text{ mm}$, so that

$$\tau_A = \frac{T}{2tA_m} = \frac{35 \text{ N}\cdot\text{m}}{2(0.005 \text{ m})(0.00200 \text{ m}^2)} = 1.75 \text{ MPa} \quad \text{Ans.}$$

And for point *B*, $t_B = 3 \text{ mm}$, and therefore

$$\tau_B = \frac{T}{2tA_m} = \frac{35 \text{ N}\cdot\text{m}}{2(0.003 \text{ m})(0.00200 \text{ m}^2)} = 2.92 \text{ MPa} \quad \text{Ans.}$$

These results are shown on elements of material located at points *A* and *B*, Fig. 5–32*e*. Note carefully how the $35\text{-N}\cdot\text{m}$ torque in Fig. 5–32*b* creates these stresses on the color-shaded faces of each element.

Angle of Twist. From the free-body diagrams in Fig. 5–32*b* and 5–32*c*, the internal torques in regions *DE* and *CD* are $35 \text{ N}\cdot\text{m}$ and $60 \text{ N}\cdot\text{m}$, respectively. Following the sign convention outlined in Sec. 5.4, these torques are both positive. Thus, Eq. 5–20 becomes

$$\begin{aligned} \phi &= \sum \frac{TL}{4A_m^2 G} \oint \frac{ds}{t} \\ &= \frac{60 \text{ N}\cdot\text{m}(0.5 \text{ m})}{4(0.00200 \text{ m}^2)^2(38(10^9) \text{ N/m}^2)} \left[2\left(\frac{57 \text{ mm}}{5 \text{ mm}}\right) + 2\left(\frac{35 \text{ mm}}{3 \text{ mm}}\right) \right] \\ &\quad + \frac{35 \text{ N}\cdot\text{m}(1.5 \text{ m})}{4(0.00200 \text{ m}^2)^2(38(10^9) \text{ N/m}^2)} \left[2\left(\frac{57 \text{ mm}}{5 \text{ mm}}\right) + 2\left(\frac{35 \text{ mm}}{3 \text{ mm}}\right) \right] \\ &= 6.29(10^{-3}) \text{ rad} \end{aligned} \quad \text{Ans.}$$

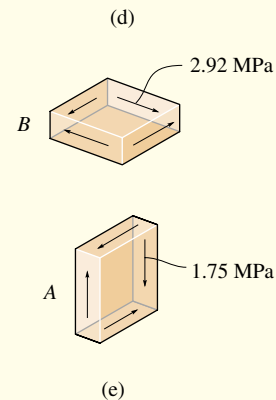
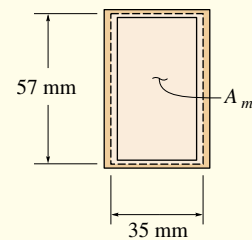
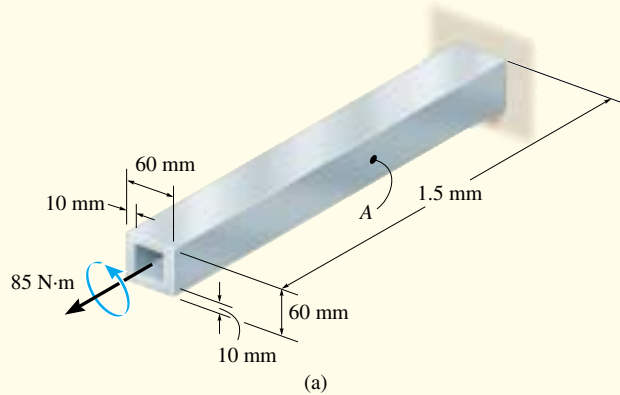


Fig. 5–32

EXAMPLE 5.16

A square aluminum tube has the dimensions shown in Fig. 5-33a. Determine the average shear stress in the tube at point A if it is subjected to a torque of 85 N·m. Also compute the angle of twist due to this loading. Take $G_{al} = 26$ GPa.



Solution

Average Shear Stress. By inspection, the internal resultant torque at the cross section where point A is located is $T = 85$ N·m. From Fig. 5-33b, the area A_m , shown shaded, is

$$A_m = (50 \text{ mm})(50 \text{ mm}) = 2500 \text{ mm}^2$$

Applying Eq. 5-18,

$$\tau_{\text{avg}} = \frac{T}{2tA_m} = \frac{85 \text{ N}\cdot\text{m} (10^3 \text{ mm/m})}{2(10 \text{ mm})(2500 \text{ mm}^2)} = 1.7 \text{ N/mm}^2 \quad \text{Ans.}$$

Since t is constant except at the corners, the average shear stress is the same at all points on the cross section. It is shown acting on an element located at point A in Fig. 5-33c. Note that τ_{avg} acts upward on the color-shaded face, since it contributes to the internal resultant torque \mathbf{T} at the section.

Angle of Twist. The angle of twist caused by T is determined from Eq. 5-20; i.e.,

$$\begin{aligned} \phi &= \frac{TL}{4A_m^2 G} \oint \frac{ds}{t} = \frac{85 \text{ N}\cdot\text{m} (10^3 \text{ mm/m})(1.5 \text{ m})(10^3 \text{ mm/m})}{4(2500 \text{ mm}^2)^2 [26(10^3 \text{ N/mm}^2)]} \oint \frac{ds}{(10 \text{ mm})} \\ &= 0.196(10^{-4}) \text{ mm}^{-1} \oint ds \end{aligned}$$

Here the integral represents the *length* around the centerline boundary of the tube, Fig. 5-33b. Thus,

$$\phi = 0.196(10^{-4}) \text{ mm}^{-1} [4(50 \text{ mm})] = 3.92(10^{-3}) \text{ rad} \quad \text{Ans.}$$

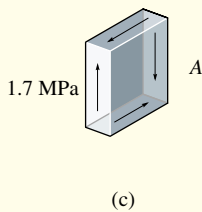
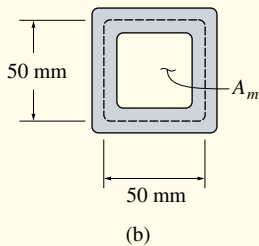
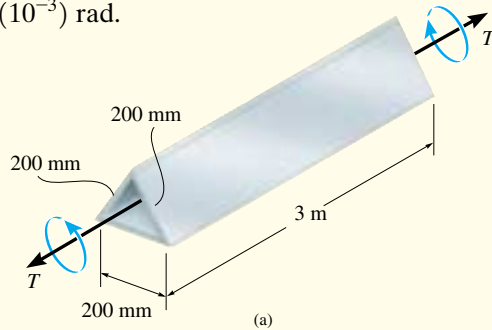


Fig. 5-33

EXAMPLE 5.17

A thin tube is made from three 5-mm-thick A-36 steel plates such that it has a cross section that is triangular as shown in Fig. 5–34*a*. Determine the maximum torque T to which it can be subjected, if the allowable shear stress is $\tau_{\text{allow}} = 90 \text{ MPa}$ and the tube is restricted to twist no more than $\phi = 2(10^{-3}) \text{ rad}$.



Solution

The area A_m is shown shaded in Fig. 5–34*b*. It is

$$\begin{aligned} A_m &= \frac{1}{2}(200 \text{ mm})(200 \text{ mm} \sin 60^\circ) \\ &= 17.32(10^3) \text{ mm}^2(10^{-6} \text{ m}^2/\text{mm}^2) = 17.32(10^{-3}) \text{ m}^2 \end{aligned}$$

The greatest average shear stress occurs at points where the tube's thickness is smallest, which is along the sides and not at the corners. Applying Eq. 5–18, with $t = 0.005 \text{ m}$, yields

$$\begin{aligned} \tau_{\text{avg}} &= \frac{T}{2tA_m}; \quad 90(10^6) \text{ N/m}^2 = \frac{T}{2(0.005 \text{ m})(17.32(10^{-3}) \text{ m}^2)} \\ T &= 15.6 \text{ kN} \cdot \text{m} \end{aligned}$$

Also, from Eq. 5–20, we have

$$\begin{aligned} \phi &= \frac{TL}{4A_m^2 G} \oint \frac{ds}{t} \\ 0.002 \text{ rad} &= \frac{T(3 \text{ m})}{4(17.32(10^{-3}) \text{ m})^2 [75(10^9) \text{ N/m}^2]} \oint \frac{ds}{(0.005 \text{ m})} \\ 300.0 &= T \oint ds \end{aligned}$$

The integral represents the sum of the dimensions along the three sides of the center-line boundary. Thus,

$$\begin{aligned} 300.0 &= T[3(0.20 \text{ m})] \\ T &= 500 \text{ N} \cdot \text{m} \end{aligned}$$

Ans.

By comparison, the application of torque is restricted due to the angle of twist.

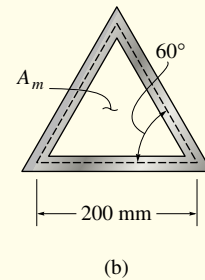


Fig. 5–34

EXAMPLE 5.18

The stepped shaft shown in Fig. 5–37*a* is supported by bearings at *A* and *B*. Determine the maximum stress in the shaft due to the applied torques. The fillet at the junction of each shaft has a radius of $r = 6$ mm.

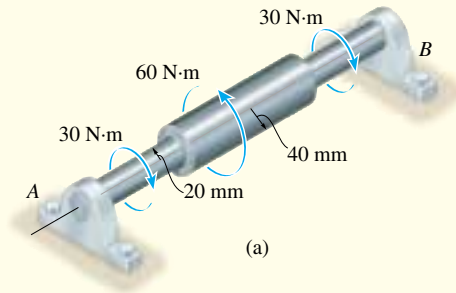
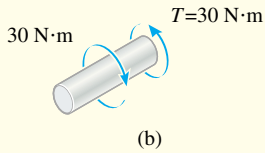


Fig. 5–37

Solution

Internal Torque. By inspection, moment equilibrium about the axis of the shaft is satisfied. Since the maximum shear stress occurs at the rooted ends of the *smaller* diameter shafts, the internal torque ($30 \text{ N}\cdot\text{m}$) can be found there by applying the method of sections, Fig. 5–37*b*.



Maximum Shear Stress. The stress-concentration factor can be determined by using Fig. 5–36. From the shaft geometry, we have

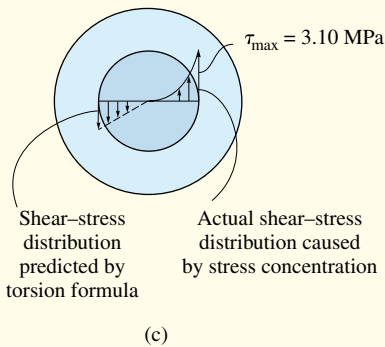
$$\frac{D}{d} = \frac{2(40 \text{ mm})}{2(20 \text{ mm})} = 2$$

$$\frac{r}{d} = \frac{6 \text{ mm}}{2(20 \text{ mm})} = 0.15$$

Thus, the value of $K = 1.3$ is obtained.

Applying Eq. 5–21, we have

$$\tau_{\max} = K \frac{Tc}{J}; \quad \tau_{\max} = 1.3 \left[\frac{30 \text{ N}\cdot\text{m}(0.020 \text{ m})}{(\pi/2)(0.020 \text{ m})^4} \right] = 3.10 \text{ MPa} \quad \text{Ans.}$$



From experimental evidence, the actual stress distribution along a radial line of the cross section at the critical section looks similar to that shown in Fig. 5–37*c*. Notice how this compares with the linear stress distribution found from the torsion formula.

EXAMPLE 5.19

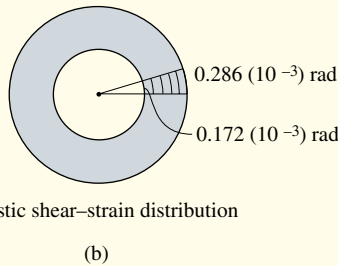
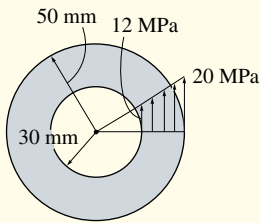
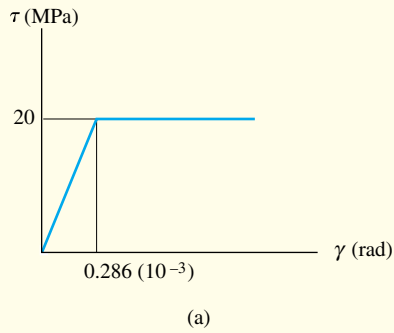
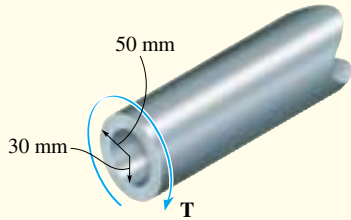


Fig. 5-42

The tubular shaft in Fig. 5-42a is made of an aluminum alloy that is assumed to have an elastic-plastic τ - γ diagram as shown. Determine (a) the maximum torque that can be applied to the shaft without causing the material to yield, (b) the maximum torque or plastic torque that can be applied to the shaft. What should the minimum shear strain at the outer radius be in order to develop a plastic torque?

Solution

Maximum Elastic Torque. We require the shear stress at the outer fiber to be 20 MPa. Using the torsion formula, we have

$$\tau_Y = \frac{T_Y c}{J}; \quad 20(10^6) \text{ N/m}^2 = \frac{T_Y(0.05 \text{ m})}{(\pi/2)[(0.05 \text{ m})^4 - (0.03 \text{ m})^4]}$$

$$T_Y = 3.42 \text{ kN} \cdot \text{m} \quad \text{Ans.}$$

The shear-stress and shear-strain distributions for this case are shown in Fig. 5-42b. The values at the tube's inner wall are obtained by proportion.

Plastic Torque. The shear-stress distribution in this case is shown in Fig. 5-42c. Application of Eq. 5-23 requires $\tau = \tau_Y$. We have

$$T_p = 2\pi \int_{0.03 \text{ m}}^{0.05 \text{ m}} [20(10^6) \text{ N/m}^2] \rho^2 d\rho = 125.66(10^6) \frac{1}{3} \rho^3 \Big|_{0.03 \text{ m}}^{0.05 \text{ m}}$$

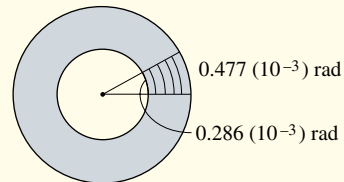
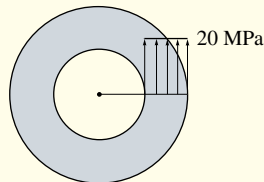
$$= 4.10 \text{ kN} \cdot \text{m} \quad \text{Ans.}$$

For this tube T_p represents a 20% increase in torque capacity compared with the elastic torque T_Y .

Outer Radius Shear Strain. The tube becomes fully plastic when the shear strain at the *inner wall* becomes $0.286(10^{-3})$ rad, as shown in Fig. 5-42c. Since the shear strain *remains linear* over the cross section, the plastic strain at the outer fibers of the tube in Fig. 5-42c is determined by proportion;

$$\frac{\gamma_o}{50 \text{ mm}} = \frac{0.286(10^{-3}) \text{ rad}}{30 \text{ mm}}$$

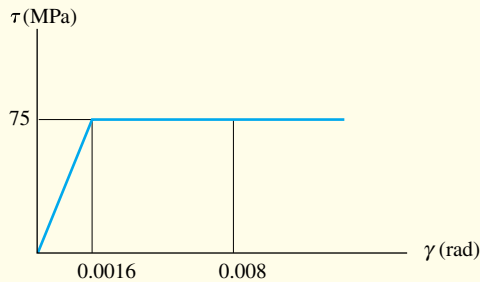
$$\gamma_o = 0.477(10^{-3}) \text{ rad} \quad \text{Ans.}$$



(c)

EXAMPLE 5.20

A solid circular shaft has a radius of 20 mm and length of 1.5 m. The material has an elastic-plastic τ - γ diagram as shown in Fig. 5-43a. Determine the torque needed to twist the shaft $\phi = 0.6$ rad.



(a)

Solution

To solve the problem, we will first obtain the shear-strain distribution, then establish the shear-stress distribution. Once this is known, the applied torque can be determined.

The maximum shear strain occurs at the surface of the shaft, $\rho = c$. Since the angle of twist is $\phi = 0.6$ rad for the entire 1.5-m length of shaft, then using Eq. 5-25 for the entire length, we have

$$\phi = \gamma \frac{L}{\rho}; \quad 0.6 = \frac{\gamma_{\max}(1.5 \text{ m})}{(0.02 \text{ m})}$$

$$\gamma_{\max} = 0.008 \text{ rad}$$

The shear-strain distribution, which always varies linearly, is shown in Fig. 5-43b. Note that yielding of the material occurs since $\gamma_{\max} > \gamma_Y = 0.0016$ rad in Fig. 5-43a. The radius of the elastic core, ρ_Y , can be obtained by proportion. From Fig. 5-43b,

$$\frac{\rho_Y}{0.0016} = \frac{0.02 \text{ m}}{0.008}$$

$$\rho_Y = 0.004 \text{ m} = 4 \text{ mm}$$

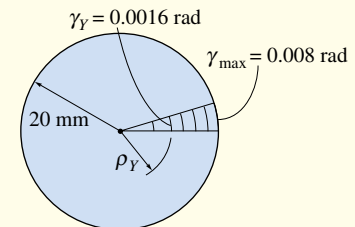
Based on the shear-strain distribution, the shear-stress distribution, plotted over a radial line segment, is shown in Fig. 5-43c. The torque can now be obtained using Eq. 5-26. Substituting in the numerical data yields

$$T = \frac{\pi \tau_Y}{6} (4c^3 - \rho_Y^3)$$

$$= \frac{\pi [75(10^6) \text{ N/m}^2]}{6} [4(0.02 \text{ m})^3 - (0.004 \text{ m})^3]$$

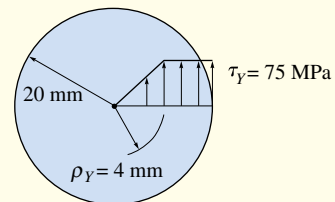
$$= 1.25 \text{ kN} \cdot \text{m}$$

Ans.



Shear-strain distribution

(b)



Shear-stress distribution

(c)

Fig. 5-43

EXAMPLE 5.21

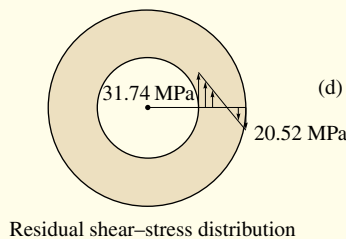
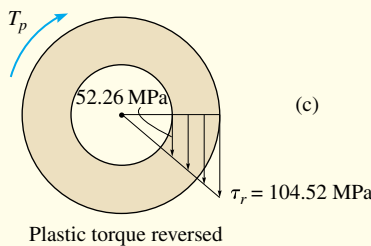
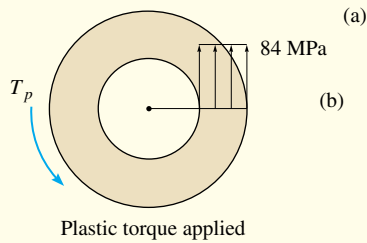
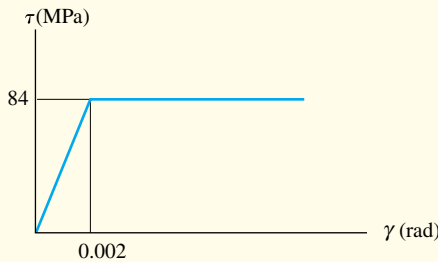
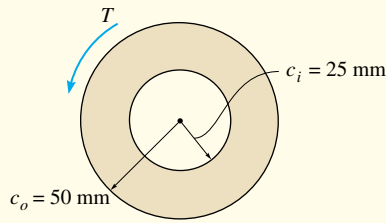


Fig. 5-46

A tube is made from a brass alloy having a length of 1.5 m and cross-sectional area shown in Fig. 5-46a. The material has an elastic-plastic τ - γ diagram, also shown in Fig. 5-46a. Determine the plastic torque T_p . What are the residual-shear-stress distribution and permanent twist of the tube that remain if T_p is removed *just after* the tube becomes fully plastic? $G = 42$ GPa.

Solution

Plastic Torque. The plastic torque T_p will strain the tube such that all the material yields. Hence the stress distribution will appear as shown in Fig. 5-46b. Applying Eq. 5-23, we have

$$T_p = 2\pi \int_{c_i}^{c_o} \tau_Y \rho^2 d\rho = \frac{2\pi}{3} \tau_Y (c_o^3 - c_i^3)$$

$$= \frac{2\pi}{3} (84 \text{ N/mm}^2) [(50 \text{ mm})^3 - (25 \text{ mm})^3] = 19.24(10^6) \text{ N}\cdot\text{mm} \quad \text{Ans.}$$

When the tube just becomes fully plastic, yielding has started at the inner radius, i.e., at $c_i = 25$ mm, $\gamma_Y = 0.002$ rad, Fig. 5-46a. The angle of twist that occurs can be determined from Eq. 5-25, which for the entire tube becomes

$$\phi_p = \gamma_Y \frac{L}{c_i} = \frac{(0.002)(1.5 \text{ m})(10^3 \text{ mm/m})}{(25 \text{ mm})} = 0.120 \text{ rad} \quad \uparrow$$

Then T_p is removed, or in effect reapplied in the opposite direction, then the “fictitious” linear shear-stress distribution shown in Fig. 5-46c must be superimposed on the one shown in Fig. 5-46b. In Fig. 5-46c, the maximum shear stress or the modulus of rupture is computed from the torsion formula

$$\tau_r = \frac{T_p c_o}{J} = \frac{19.24(10^6) \text{ N}\cdot\text{mm} (50 \text{ mm})}{(\pi/2)[(50 \text{ mm})^4 - (25 \text{ mm})^4]} = 104.52 \text{ N/mm}^2 = 104.52 \text{ MPa}$$

Also, at the inner wall of the tube the shear stress is

$$\tau_i = (104.52 \text{ MPa}) \left(\frac{25 \text{ mm}}{50 \text{ mm}} \right) = 52.26 \text{ MPa}$$

From Fig. 5-46a, $G = \tau_Y/\gamma_Y = 84 \text{ N/mm}^2/(0.002 \text{ rad}) = 42(10^3) \text{ MPa}$, so that the corresponding angle of twist ϕ'_p upon removal of T_p is therefore

$$\phi'_p = \frac{T_p L}{JG} = \frac{19.24(10^6) \text{ N}\cdot\text{mm} (1.5 \text{ m})(10^3 \text{ mm/m})}{(\pi/2)[(50 \text{ mm})^4 - (25 \text{ mm})^4] 42(10^3) \text{ N/mm}^2} = 0.0747 \text{ rad} \quad \downarrow$$

The resulting residual-shear-stress distribution is therefore shown in Fig. 5-46d. The permanent rotation of the tube after T_p is removed is

$$\downarrow + \quad \phi = 0.120 - 0.0747 = 0.0453 \text{ rad} \quad \uparrow \quad \text{Ans.}$$